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THE

ELEMENT

OF

G E O M E T R Y.

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THE ELEMENT OF GEOMETRY.

BOOK I.

DEFINITIONS AND REMARKS.

I.

THE word Geometry is from the Greek, it means the measurement of the earth, and is the name of the science which treats of space, as defined by shape or extent.

II.

Two portions of space the same in shape, may be called similar.

III.

The same in extent, equal.

IV.

The same in shape and extent, equivalent.

V.

A portion of space extending in length, breadth, and thickness, may be called a solid.

VI.

In length and breadth, without reference to thickness, a surface.

VII.

In length without reference to breadth or thickness, a line.

VIII.

Place, without reference to extent, may be called a point.

IX.

A point is considered incapable of division, and not to occupy space.

X.

A solid is bounded by a surface, or surfaces.

XI.

A surface is bounded by a line, or lines.

XII.

A line is bounded by points.

XIII.

The intersection of two lines is a point.

XIV.

If a line is extended, its parts will approach more nearly to the same direction, and if it is extended till its extremities are at the greatest distance which the length of the line will permit, every part will then be in the same direction, and it may be called a stretched line, or straight line.

XV.

A straight line may be supposed to be drawn from any point, in any direction, and to any distance.

XVI.

If a line that is not straight between two points be straightened, part of the line will be drawn beyond one of the points, and the remainder will extend from one to the other; therefore, a straight line is the shortest line that can be drawn between two points.

XVII.

To continue a straight line in the original direction, may be called producing the line.

XVIII.

To draw a straight line from one point to another, may be called joining the points.

XIX.

Two magnitudes, which being compared exactly fill the same space, may be said to coincide.

XX.

If one straight line is applied to another, they will coincide, except so far as one extends beyond the other, and if they are produced they will still coincide.

XXI.

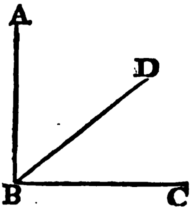
If a straight line be drawn between two points, any other straight line between the same points will coincide with the first line.

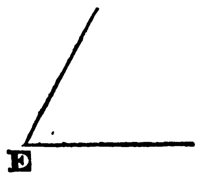
XXII.

Two straight lines, in different directions from the same point, may be said to make an angle.

XXIII.

An angle may be named by a letter at the angular point, as E; or by letters distinguishing the lines which make the angle, as A B C,





A B D, or C B D, the letter at the angular point being between the other two.

XXIV.

The angle is determined by the difference in direction between the lines, and is not varied by their length, or by producing them.

XXV.

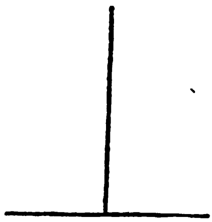
If two straight lines which make an angle are produced, the distance between their extremities will be increased, and the more they are produced, the more will it be increased.

XXVI.

If two straight lines meet and are produced, they will either coincide, or not coincide. If they coincide, and are produced, they will still coincide, and cannot enclose a space. If they do not coincide, they will make an angle, and cannot enclose a space. Therefore, two straight lines cannot enclose a space.

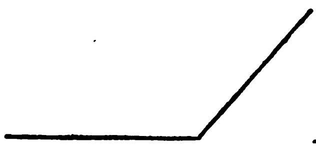
XXVII.

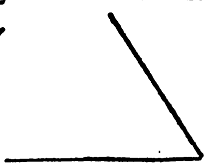
If a straight line standing on another straight line, makes the angles on each side equal, each of them may be called a right angle, and the lines may be said to be perpendicular to each other.



XXVIII.

An angle greater than a right angle, may be called an obtuse angle.





XXIX.

An angle less than a right angle, may be called an acute angle.

XXX.

An acute or obtuse angle, may be called an oblique angle.

XXXI.

A portion of space enclosed by one or more boundaries, may be called a figure.

XXXII.

A figure enclosed by straight lines, may be called a rectilineal figure.

XXXIII.

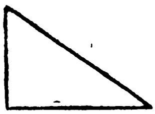
In describing a rectilineal figure, the two first lines make an angle, every succeeding line makes an angle with the preceding, and the last line makes an angle with the preceding, and with the first line. Therefore, in every rectilineal figure, the number of angles is equal to the number of sides.

XXXIV.

A figure enclosed by three straight lines may be called a triangle.

XXXV.

A triangle containing a right angle may be called a right angled triangle.

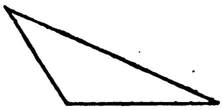


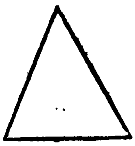
XXXVI.

The side opposite the right angle may be called the hypothenuse, and the other two sides the legs. One of the legs may be called the base, and the other the perpendicular.

XXXVII.

A triangle containing an obtuse angle may be called an obtuse angled triangle.





XXXVIII.

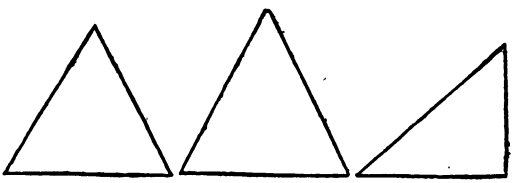
A triangle containing three acute angles may be called an acute angled triangle.

XXXIX.

An obtuse, or acute angled triangle, may be called an oblique angled triangle.

XL.

A triangle having three equal sides may be called an equilateral triangle.



XLI.

A triangle having two equal sides may be called an isosceles triangle.

XLII.

A triangle having the three sides unequal may be called a scalene triangle.

XLIII.

If a straight line move on two straight lines which make an angle, the surface thus described may be called a plane.

XLIV.

A plane may be supposed to be described on any two straight lines which make an angle, and enlarged to any extent.

XLV.

To enlarge a plane on the original lines may be called extending the plane.

XLVI.

Because a plane may be described on any two sides of a rectilineal triangle, and extended to the remaining side; therefore a rectilineal triangle is a plane figure.

XLVII.

A straight line may be applied to any point in a plane, and moved in the plane till it coincides with any other point therein. Therefore, if any two points be taken in a plane, the straight line between them lies wholly in the plane.

XLVIII.

If a part of a straight line is in a plane, the remainder is also in the plane; and if the line is produced, and the plane extended, the line remains in the plane.

XLIX.

If one plane be applied to another they will coincide, except so far as one extends beyond the other; and if they are extended, they will still coincide.

L.

A straight line which revolves in a plane about one of its own extremities till it returns to its original position, may be called a radius.

LI.

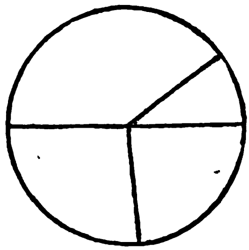
The point of revolution may be called a centre.

LII.

The boundary described by the opposite extremity of the radius may be called a circumference.

LIII.

And the figure thus described may be called a circle.



LIV.

Because the radius measures all straight lines from the centre to the circumference of the circle, therefore they are all equal.

LV.

A straight line passing through the centre of a circle, and terminated at either extremity by the circumference, may be called a diameter.

LVI.

A part of the circumference of a circle may be called an arc.

LVII.

If a line or figure is divided into two equal parts, it may be said to be bisected.

DEFINITIONS

Of terms which are used in Geometry.

I. A request that cannot reasonably be denied, may be called a postulate.

II. That which is proposed to be done or proved, may be called a proposition.

III. A proposition to be done may be called a problem.

IV. A proposition to be proved may be called a theorem.

V. A theorem, whose evidence arises immediately from the consideration of the terms in which it is stated, may be called an axiom.

VI. A proposition to assist in doing or proving another, may be called a lemma.

VII. A supposition made in stating or performing a proposition, may be called an hypothesis.

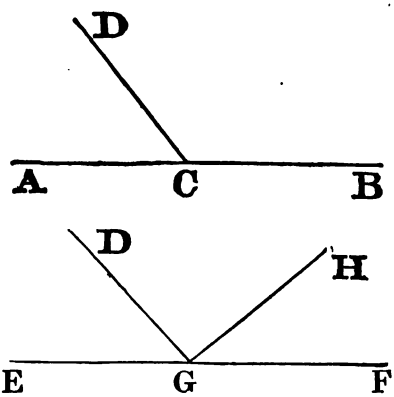
VIII. An inference which follows from one or more propositions, may be called a corollary.

IX. A remark upon one or more propositions, tending to show their connection, their restriction, their extension, or the manner of their application, may be called a scholium.

PROPOSITION I. THEOREM.

If from two points in two straight lines, two other straight lines be drawn, the sum of the angles made by one of the lines is equal to the sum of the angles made by the other.

From the points C and G in the straight lines AB and EF, let the straight lines CD and GH be drawn, the sum of the angles ACD, BCD is equal to the sum of the angles EGH, FGH. Let AB be applied to EF, so that C coincides with G; AB will coincide with EF (Def. 20.), and let CD fall as GD, then the angles ACD, BCD will coincide with the angles EGD and DGF. Because the angles EGD and DGF are composed of the three angles EGD, DGH, FGH, and the angles EGH, FGH also composed of the same three angles, EGD and DGF are together equal to EGH and FGH. But the angles EGD and DGF, coincide with the angles ACD and BCD, and therefore the angles ACD and BCD are together equal to the angles EGH and FGH. Therefore, if from two points in two straight lines, two other straight lines be drawn, the sum of the angles made by one of the lines is equal to the sum of the angles made by the other. Which was to be proved.



Corollary I. As the straight lines CD and GH may be drawn on either side of the straight lines AB and EF; therefore the sum of the angles on one side of a straight line is equal to the sum of the angles on the other side.

Cor. II. Because a right angle is equal to half the sum of all the angles on one side of a straight line (Def. 27.); therefore all right angles are equal.

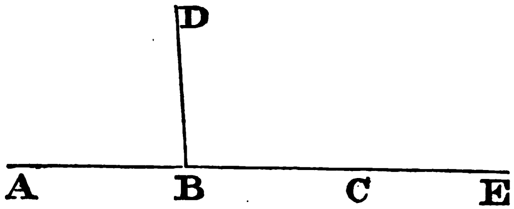
Cor. III. The angles which one straight line makes with another on the same side of it, are together equal to two right angles.

PROP. II. THEOR.

If, at a point in a straight line, two other straight lines on the opposite sides of it make the adjacent angles together equal to two right angles, these two straight lines shall be in the same straight line.

At the point B in the straight line BD, let the straight lines AB, BC make the angles ABD, CBD together equal to two right angles; AB

and BC are in the same straight line. Produce AB to E, and because AE is a straight line, the angles ABD and DBE are together equal to two right angles (3 Cor. I. Prop. I.). But ABD and CBD are equal to two right angles by the hypothesis; therefore ABD and DBE are

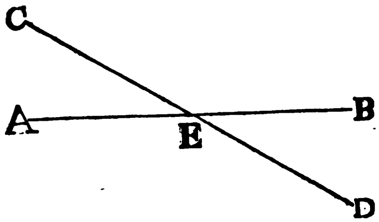


equal to ABD and CBD. Take away the common angle ABD, and the remaining angle DBE is equal to the remaining angle CBD; and therefore BE coincides with BC. Therefore, if from a point in a straight line, two other straight lines on the opposite sides of it make the adjacent angles together equal to two right angles, these two straight lines shall be in the same straight line. Which was to be proved.

PROP. III. THEOR.

If two straight lines cut one another, the vertical or opposite angles shall be equal.

Let the two straight lines AB, CD cut one another in the point E, the angle AEC shall be equal to the angle BED, and the angle AED to the angle CEB. Because the straight line AE makes with CD the angles AEC, AED; these angles are together equal to two right angles (3 Cor. I. Prop. I.). Again, because the straight line DE makes with AB the angles AED, BED; these also are together equal to two right angles (3 Cor. I. Prop. I.): and AEC, AED have been shewn to be equal to two right angles; therefore the angles AEC, AED



are equal to the angles AED, BED. Take away the common angle AED, and the remaining angle AEC is equal to the remaining angle BED. In the same manner it can be shewn, that the angles AED, BEC are equal. Therefore, if two straight lines cut one another, the vertical or opposite angles shall be equal. Which was to be proved.

Cor. I. If two straight lines cut one another, the four angles which they make at the point where they cut, are together equal to four right angles.

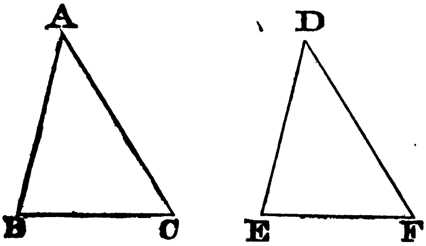
Cor. II. All the angles made by any number of straight lines, which meet in one point, are together equal to four right angles.

PROP. IV. THEOR.

If two triangles have two sides of the one equal to two sides of the other, each to each; and have likewise the angles contained by those sides equal to one another, they shall likewise have their bases, or third sides, equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, viz. those to which the equal sides are opposite.

Let ABC, DEF be two triangles which have the two sides AB, AC equal to the two sides DE, DF, each to each, viz. AB to DE, and AC to DF; and the angle BAC equal to the angle EDF, the base BC shall be equal to the base EF; and the triangle ABC to the triangle DEF; and the other angles, to which the equal sides are opposite, shall be equal, each to each, viz. the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

For, if the triangle ABC be applied to DEF, so that the point A may be on D, and the straight line AB upon DE, the point B shall



coincide with the point E, because AB is equal to DE; and AB coinciding with DE, AC shall coincide with DF, because the angle BAC is equal to the angle EDF; wherefore, also, the point C shall coincide with the point F, because the straight line AC is equal to DF; but the point B coincides with the point E; wherefore, the base BC shall coincide with the base EF (Def. 21.). Wherefore, the whole triangle ABC shall coincide with the whole triangle DEF, and be equal to it; and the other angles of the one shall coincide with the remaining angles of the other, and be equal to them, viz. the angle ABC to the angle DEF, and the angle ACB to DFE. Therefore, if two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to one another; their bases shall likewise be equal, and the triangles be equal, and their other angles to which the equal sides are opposite shall be equal, each to each. Which was to be proved.

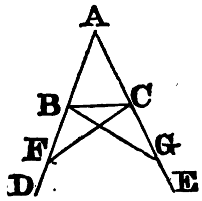
PROP. V. THEOR.

The angles at the base of an isosceles triangle are equal to one another; and, if the equal sides be produced, the angles upon the other side of the base shall be equal.

Let ABC be an isosceles triangle, of which the side AB is equal to

AC, and let the straight lines AB, AC, be produced to D and E, the angle ABC shall be equal to ACB, and the angle CBD to the angle BCE. In BD take any point F, and make AG equal to AF, and join BG, CF.

Because AF is equal to AG, and AB to AC, the two sides AF, AC are equal to the two sides AG and AB, each to each; and they contain the angle FAG common to the two triangles ACF, ABG; therefore (4. 1.), the triangle ACF is equivalent to the triangle ABG; and therefore the angle AFC is equal to the angle AGB: and because the whole AF is equal to the whole AG, of which the parts AB, AC are equal; the remainder BF shall be equal to the remainder CG; but CF is equal (4. 1.) to BG; therefore, the two sides BF, CF are equal to the two CG, BG, each to each; and the angle BFC is equal to the angle CGB; wherefore (4. 1.) the triangles are equivalent: therefore, the angle FBC is equal to the angle GCB, and the angle BCF to the angle CBG; and, since it has been shewn that the whole angle ABG is equal to the whole angle ACF, the parts of which, the angles CBG, BCF are also equal; the remaining angle ABC is therefore equal to the remaining angle ACB, which are the angles at the base of the triangle ABC; and it has also been proved, that the angle FBC is equal to the angle GCB, which are the angles upon the other side of the base. Therefore, the angles at the base of an isosceles triangle are equal to one another; and, if the equal sides be produced, the angles upon the other side of the base shall be equal. Which was to be proved.

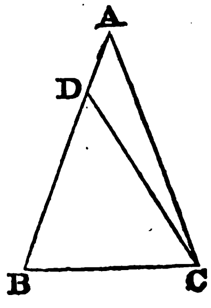


COR. Hence, every equilateral triangle is also equiangular.

PROP. VI. THEOR.

If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another.

Let ABC be a triangle, having the angle ABC equal to the angle ACB; the side AB is also equal to the side AC. For, if AB be not



equal to AC, one of them is greater than the other; let AB be the greater, and from it cut off DB equal to AC the less, and join DC; therefore, because in the triangles DBC, ACB, DB is equal to AC, and BC common to both, the two sides BD, BC are equal to the two sides AC, BC, each to each; and the angle DBC is equal to the angle ACB; therefore (4. 1.), the base DC is equal to the base AB, and the triangle CBD is equivalent to the triangle ACB, the less to the greater; which is absurd. Therefore, AB is not unequal to AC, that is, it is equal to it. Wherefore, if two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another. Which was to be proved.

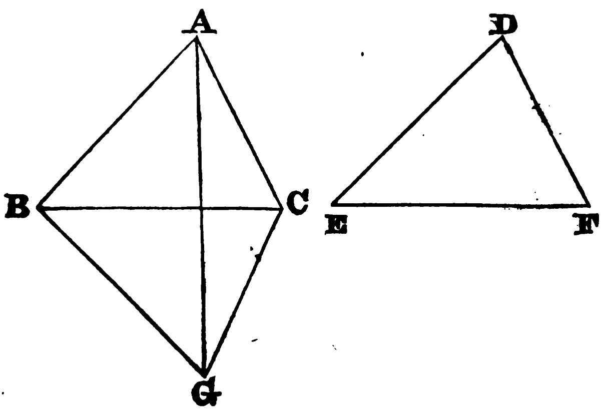
Cor. Hence, every equiangular triangle is also equilateral.

PROP. VII. THEOR.

If two triangles have the three sides of the one equal to the three sides of the other, each to each; the angles opposite the equal sides are also equal.

Let the two triangles ABC, DEF, have the three sides equal, that is, AB equal to DE, AC to DF, and BC to EF. The angles opposite the equal sides shall also be equal; that is, the angle BAC to the angle EDF, ABC to DEF, and ACB to DFE.

Let the triangles be joined by the longest equal sides BC and EF, having the remaining sides GB equal to DE, and GC to DF, and join AG. Because in the triangle ABG the side AB is equal to the side BG by the hypothesis, the angle BAG is equal to the angle BGA (5. 1.); for the same reason, in the triangle ACG the angle CAG is equal to the angle CGA; and therefore, the two angles BAG and



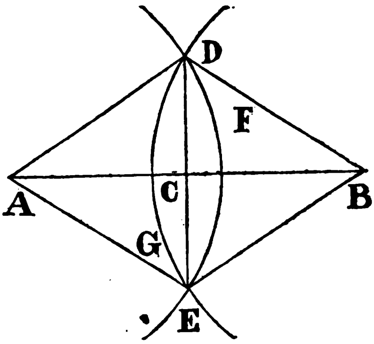
CAG, or BAC, are equal to the two angles BGA and CGA, or BGC; then the triangles ABC, GBC have the two AB, AC equal to the two sides GB, GC, and the contained angle BAC equal to the contained

angle BGC. And, therefore, the triangles are equivalent (4. 1.); and the angle ABC equal to the angle GBC, or DEF, and the angle ACB to the angle GCB, or DFE. Therefore, if, &c. Q. E. D.

PROP. VIII. PROBLEM.

To bisect a given finite straight line.

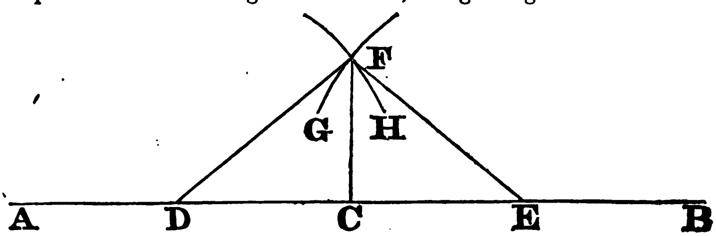
Let AB be the given finite straight line; it is required to bisect it. From A, with a radius (Def. 50.) AF greater than half AB, describe the arc DFE; from B, with the same radius, describe the arc DGE, intersecting DFE in D, E, and join DE, intersecting AB at C; AB is bisected at C. Join AD, BD, BE, AE. Because the two triangles ADE, BDE, have the two sides AD, AE, equal to the two sides BD, BE, and the third side DE common; therefore (7. 1.), the angle DAE is equal to the angle DBE, and therefore (4. 1.), the two triangles ADE, DBE are equivalent, and the angle ADE or ADC is equal to the angle BDE or BDC. And in the triangles ADC, BDC, the side AD is equal to the side BD, and DC is common, and the angle ADC contained by them, has been proved to be equal to the angle BDC; therefore (4. 1.), the triangles are equivalent, and the third side AC is equal to the third side BC, and the straight line AB is divided into two equal parts in the point C. Which was to be done.



PROP. IX. PROB.

To draw a straight line at right angles to a given straight line, from a given point in the same.

Let AB be the given straight line, and C the given point in it; it is required to draw a straight line from C, at right angles to AB. Take



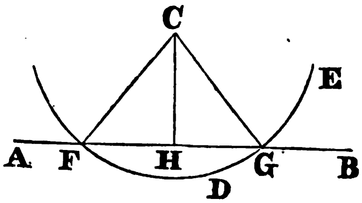
any point D in AC, and make CE equal to CD; from D, with any radius greater than DC, describe the arc FH, and from E with the

same radius, describe the arc FG, intersecting FH in F, and join CF; CF is perpendicular to AB. Join DF, EF. Because the triangles DFC, EFC, have the sides DF, CD equal to EF, CE, and the third side CF common; therefore (7. 1.), the angles CDF, CEF are equal; and therefore (4. 1.), the two triangles DFC, EFC are equivalent; and the angle DCF is equal to the angle BCF, and therefore (Def. 27.) CF is perpendicular to DE, or AB. Wherefore, from the given point C, in the given straight line AB, CF has been drawn at right angles to AB. Which was to be done.

PROP. X. PROB.

To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.

Let AB be the given straight line, which may be produced to any length both ways, and let C be a point without it. It is required to draw a straight line perpendicular to AB from the point C. Take any point D, upon the other side of AB, and from C as a centre (Def. 50.), with a radius CD, describe the arc FDG, meeting AB in FG; and bisect (8. 1.) FG in H, and join CF, CH, CG; the straight line CH, drawn from the given point C, is perpendicular to the given straight line AB. Because FH is equal to GH, and CH common to the two triangles FHC, GHC, the two sides FH, CH are equal to the two sides GH, CH,

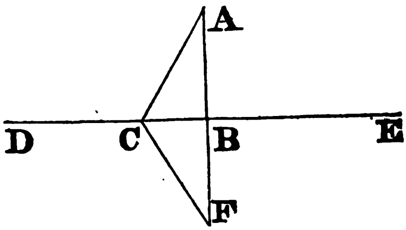


each to each; and the base CF is equal (Def. 54.) to the base CG; therefore, the angle CHF is equal (7. 1.) to the angle CHG, and they are adjacent angles; but when a straight line, standing on a straight line, makes the adjacent angles equal to one another, each of them is a right angle, and the straight line which stands upon the other, is called a perpendicular to it (Def. 27.); therefore, from the given point C, a perpendicular CH has been drawn to the given straight line AB. Which was to be done.

PROP. XI. THEOR.

From a point without a straight line, only one perpendicular can be drawn to the line.

Let DE be a straight line, A a point without it, and AB perpendicular to DE: no other perpendicular can be drawn from A to DE. If possible, let AC be also perpendicular. Produce AB to F, make BF equal to AB, and join FC. Because the angle ABC is a right angle, the angle FBC is also a right angle (Cor. 3. 1.) and the two triangles ABC, FBC have the side AB equal to BF, BC common, and the contained angle ABC, equal to the contained angle FBC; therefore, the triangles

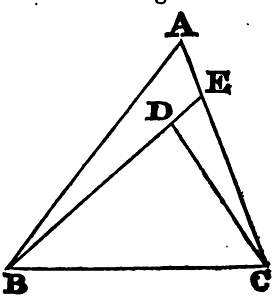


are equivalent (4. 1.), and the angle ACB is equal to the angle FCB; but ACB is a right angle by the hypothesis, therefore FCB is also a right angle; but when at a point in a straight line two other straight lines make the adjacent angles equal to two right angles, these two straight lines shall be in the same straight line (2. 1.); therefore ACF is a straight line; and between the points A and F there are two straight lines which do not coincide; but (Def. 21.) that is impossible; and, therefore, ACB is not a right angle. Therefore, From, &c. Q. E. D.

PROP. XII. THEOR.

If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle.

Let the two straight lines BD, CD be drawn from B, C, the ends of the side BC of the triangle ABC, to the point D within it. BD and DC are less than the other two sides BA, AC of the triangle.

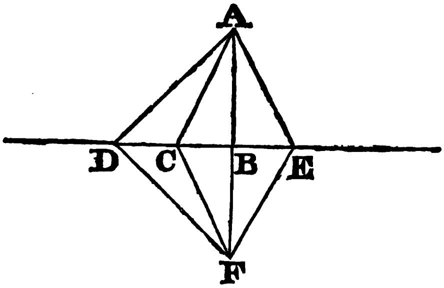


Produce BD to E; and because two sides of a triangle are greater than the third side, the two sides BA, AE of the triangle ABE are greater than BE: to each of these add EC, therefore the sides BA, AC are greater than BE, EC: again, because the two sides CE, ED of the triangle CED are greater than CD, add DB to each of these; therefore the sides CE, EB are greater than CD, DB: but it has been shewn that BA, AC are greater than BE, EC; much more then are BA, AC greater than BD, DC. Therefore, If from the ends, &c. Q. E. D.

PROP. XIII. THEOR.

If from a point without a straight line a perpendicular be drawn to the line, and different straight lines making oblique angles with the straight line, the perpendicular will be shorter than either of the oblique lines. Two obliques equally distant from the perpendicular will be equal, and an oblique farther from the perpendicular, will be longer than one nearer thereto.

Let A be a point without the straight line DE, AB perpendicular to DE, AC and AE obliques, equidistant from AB, and AD farther than AC from AB. AB will be less than AE, AC, or AD; AC and AE will be equal, and AD greater than AC. Produce AB to F, make BF equal to AB, and join AC, AD, AE, FC, and FD. Because ABC is a right angle, FBC is also a right angle (Cor. 3. 1.), and the tri-



angles ABC, FBC have the side AB equal to the side BF, the side CB common, and the contained angle ABC equal to the contained angle FBC; therefore the triangles are equivalent (4. 1.), and the side AC is equal to the side FC. But AF is a straight line, and therefore it is less than ACF (Def. 16.); and therefore AB, the half of AF, is less than AC, the half of ACF; therefore the perpendicular is shorter than any oblique.

Because the triangles ABC, ABE have the side BC equal to the side BE, the side AB common, and the contained angle ABC equal to the contained angle ABE, being both right angles by the hypothesis, (Cor. 2. 1.), the triangles are equivalent, and the side AC is equal to the side AE; therefore two obliques equidistant from the perpendicular, are equal.

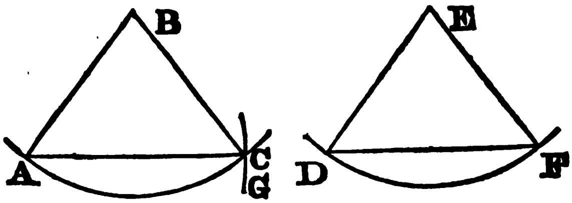
In the same way it may be shewn, that the triangle ABD is equivalent to the triangle FBD, and the side AD equal to the side FD. Then the straight lines AC, FC are drawn to the point C, within the triangle ADF; therefore, they are less than AD and FD (13. 1.), and therefore AC, the half of AC, FC, is less than AD, the half of AD, FD; and therefore the oblique farther from the perpendicular, will be greater than one nearer thereto. Therefore, If, &c. Q. E. D.

Cor. A perpendicular from a point to a straight line, measures the shortest distance between the point and the line.

PROP. XIV. PROB.

At a given point in a given straight line, to make an angle equal to a given angle.

Let B be the given point in the given straight line AB, and let DEF be the given angle; it is required to make at B an angle with the line



AB, equal to DEF. From E, with any radius ED, describe the arc DF, and from B, with the same radius, describe the arc AC; join DF, and from A, with the radius DF, describe the arc CG, intersecting the arc AC at C, and join BC; the angle ABC is equal to the angle DEF.

Join AC; because the triangles ABC, DEF, have two sides AB, BC equal to the two sides DE, EF, and the third side AC equal to the third side DF; therefore the angle ABC is equal (7. 1.) to the angle DEF. Therefore, at the given point B, in the given straight line AB, the angle ABC is made equal to the given angle DEF. Which was to be done.

THE

ELEMENT

OF

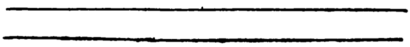
GEOMETRY.

BOOK. II.

DEFINITIONS.

I.

Two straight lines in the same direction from different points, may be called parallel lines.



II.

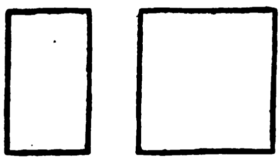
A plane figure enclosed by four straight lines, may be called a quadrangle.

III.

A quadrangle, whose opposite sides are parallel, may be called a parallelogram.

IV.

A parallelogram, whose angles are right angles, may be called a rectangle.

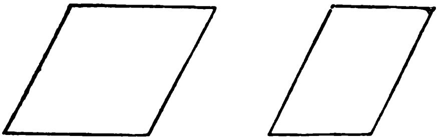


V.

A rectangle, whose sides are equal, may be called a square.

VI.

A parallelogram, whose sides are equal and its angles oblique, may be called a rhombus.



VII.

If the sides are unequal and the angles oblique, a rhomboid.

VIII.

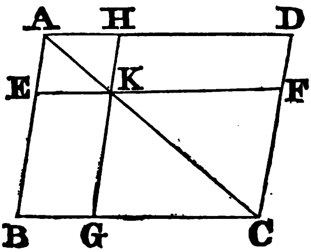
A quadrangle that is not a parallelogram, may be called a trapezium.

IX.

A straight line joining the two opposite angles of a parallelogram, may be called a diagonal.

X.

Let ABCD be a parallelogram, AC a diagonal. Through K, any point in AC, draw EF parallel to AD; and also GH parallel to AB. A parallelogram may be named by the letters at two opposite angles, as AC.



XI.

The parallelograms through which the diagonal passes, may be said to be about the diagonal, as EH, GF.

XII.

The remaining parallelograms, as EG, HF, may be called the complements.

XIII.

The complements, and one of the parallelograms about the diagonal, may be said to make a gnomon.

XIV.

A gnomon may be expressed by the letters at the opposite angles, as AGF.

XV.

A rectangle may be said to be contained by the two lines which contain one of the angles.

XVI.

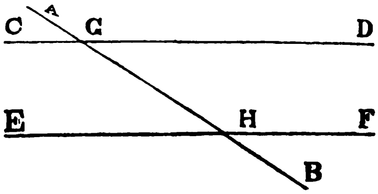
A perpendicular drawn from an angle of a triangle or parallelogram, to the base or base produced, may be called the altitude of the triangle, or parallelogram.

PROPOSITION I. THEOREM.

If a straight line falling on two other straight lines, makes an angle with one of the lines equal to an angle which it makes with the other, the remaining angles which it makes with the first line, are equal to the remaining angles which it makes with the other line, each to each.

Let the straight line AB, falling on the straight lines CD, EF, make the angle AGD equal to the angle GHF; the angle AGC will be equal to the angle GHE, the angle EHB to CGH, and BHF to DGH.

Because AG and CD are straight lines, the angles AGD and AGC are together equal (3 cor. 1. 1.) to two right angles. For the same reason, the angles GHF and GHE are together equal to two right angles; take away the equal angles AGD and GHF, and the remainder AGC is equal to the remainder GHE: and because AGC and DGH are vertical angles, they are equal (3. 1.); but AGC has been proved to be equal to GHE, and therefore DGH is equal to GHE; and because GHE and FHB are vertical angles, they are equal (3. 1.); and therefore DGH is equal to BHF: and in the same way it may be prov-



ed that EHB is equal to CGH. Therefore, if a straight line falling on two other straight lines, makes an angle with one of the lines equal to an angle which it makes with the other, the remaining angles which it makes with the first line, are equal to the remaining angles which it makes with the other line, each to each. Which was to be proved.

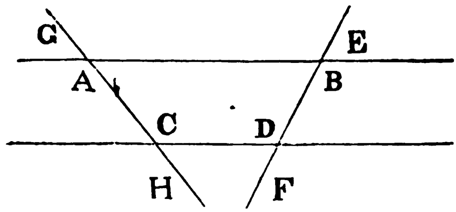
Cor. The similarly placed angles AGD and GHF, and their verticals CGH and BHE, may be called corresponding angles. Likewise AGC and GHE, and their verticals DGH and BHF.

PROP. II. THEOR.

If a straight line, falling on two other straight lines in the same plane, makes the corresponding angles equal, any other straight line falling on those two straight lines, will also make the corresponding angles equal.

Let the straight line EF fall on the straight lines AB and CD, and make the corresponding angles equal; any other straight line, as GH, will also make the corresponding angles equal.

Because the angles which EF makes with AB, are equal to the angles which it makes with CD, the difference in direction between



EF and CD, is equal to the difference in direction between EF and AB; and therefore CD is in the same direction with AB; but every part of the same straight line is in the same direction (Def. 14. 1.), and therefore every part of AB is in the same direction with every part of CD; and because AB and CD are in the same direction, and GH falls on them, the difference in direction between GH and AB is equal to the difference in direction between GH and CD; and therefore the angles which GH makes with CD, are equal to the corresponding angles which it makes with AB. Therefore, if a straight line, falling on two other straight lines, makes the corresponding angles equal, any other straight line falling on these two straight lines, will also make the corresponding angles equal. Which was to be proved.

Cor. I. If a straight line falling on two other straight lines in the same plane, makes the corresponding angles equal; those two straight lines are parallel.

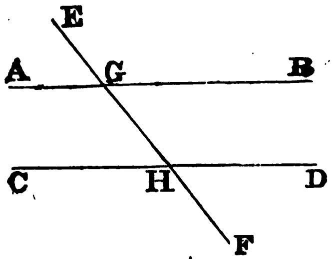
Cor. II. If a straight line falling on two other straight lines in the same plane, makes the two interior angles on the same side together equal to two right angles; those two straight lines are parallel.

PROP. III. THEOR.

If a straight line fall on two parallel lines, it makes the corresponding angles equal.

Let the straight line EF fall on the two parallel lines AB, CD; it makes the corresponding angles equal.

Because AB is parallel to CD, it is in the same direction (Def. 1. 2.); and therefore the difference in direction between AB and EF is equal to the difference in direction between CD and EF, and therefore the angles which EF makes with AB, are equal to the corresponding angles which it makes with CD.



Cor. I. Because the angle EGB is equal to the angle GHD, the angles EGB and BGH are together equal to the angles GHD and BGH; but EGB and BGH are together equal to two right angles; therefore GHD and BGH are together equal to two right angles. Therefore,

if a straight line fall on two parallel lines, it makes the two interior angles on the same side together equal to two right angles.

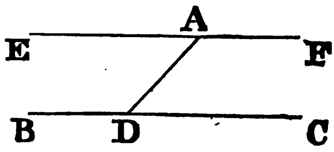
Cor. II. Straight lines that are parallel to the same straight line, are parallel to one another.

PROP. IV. PROB.

To draw a straight line through a given point parallel to a given straight line.

Let A be the given point, and BC the given straight line; it is required to draw a straight line through the point A parallel to the straight line BC. In BC take any point D, and join AD; and at the point A in the straight line AD, make (14. 1.) the angle DAE equal to the angle ADC; and produce the straight line AE to F.

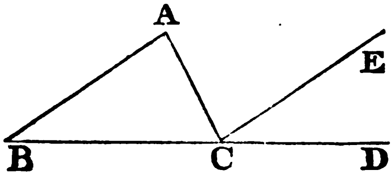
Because the straight line AD, falling on the two straight lines BC, EF, makes an angle DAE, with one of the lines EF, equal to an angle ADC, which it makes with the other line BC, the remaining angles which AD makes with EF, are equal to the remaining angles it makes with BC (1. 2.); and any other straight line falling on EF and BC, will also make the corresponding angles equal (2. 2.); and therefore EF is in the same direction as BC, or is parallel to it. Therefore, the straight line EAF is drawn through the given point A, parallel to the given point BC. Which was to be done.



PROP. V. THEOR.

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are equal to two right angles.

Let ABC be a triangle, and let any of its sides BC be produced to D; the exterior angle ACD is equal to the two interior and opposite angles CAB, ABC; and the three interior angles of the triangle, viz. ABC, BCA, CAB, are together equal to two right angles.

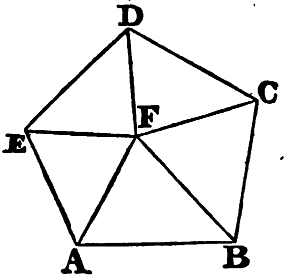


Through the point C draw CE parallel (4. 2.) to the straight line AB; and because AB is parallel to CE, and AC meets them, the corresponding angles BAC, ACE are equal (3. 2.). Again, because AB

is parallel to CE, and BD falls upon them, the corresponding angles ABC, ECD are equal; but the angle ACE was shown to be equal to the angle BAC; therefore the whole exterior angle ACD is equal to the two interior and opposite angles CAB, ABC: to these equals add the angle ACB, and the angles ACD, ACB are equal to the three angles CBA, BAC, ACB; but the angles ACD, ACB are equal (3 Cor. 1. 1.) to two right angles: therefore, also, the angles CBA, BAC, ACB, are equal to two right angles. Wherefore, if a side of a triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are equal to two right angles. Which was to be proved.

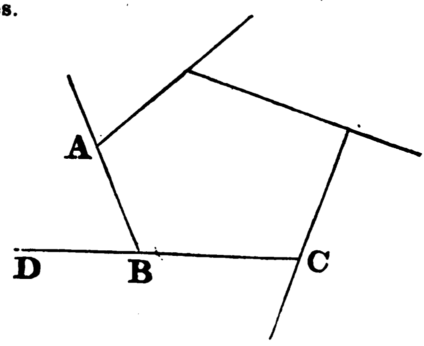
Cor. I. All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

For any rectilineal figure ABCDE can be divided into as many triangles as the figure has sides, by drawing straight lines from a point F within the figure to each of its angles. And, by the preceding proposition, all the angles of these triangles are equal to twice as many right angles as there are triangles, that is, as there are sides of the figure; and the same angles are equal to the angles of the figure, together with the angles at the point F which is the common vertex of the triangles; that is (Cor. 2. 3. 1.), together with four right angles. Therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.



Cor. II. All the exterior angles of any rectilineal figure are together equal to four right angles.

Because every interior angle ABC with its adjacent exterior ABD is equal (Cor. 3. 1. 1.) to two right angles; therefore all the interior together with all the exterior angles of the figure, are equal to twice as many right angles as there are sides of the figure, that is, by the foregoing Corollary, they are equal to all the interior angles of the figure, together with four right angles: therefore all the exterior angles are equal to four right angles.

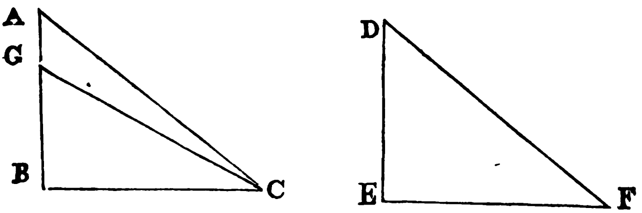


PROP. VI. THEOR.

If two triangles have two angles of the one equal to two angles of the other, each to each; and one side equal to one side, the triangles shall be equivalent.

Let ABC, DEF be two triangles which have the angles ABC, BCA, equal to the angles DEF, EFD, viz. ABC to DEF, and BCA to EFD; also let BC be equal to EF, the triangles shall be equivalent.

Because the three angles in each of the triangles ABC, DEF, are equal to two right angles (5. 2.), they are equal to each other; take away the angles ABC, BCA, equal to the angles DEF, EFD, and the remaining angle BAC will be equal to the remaining angle EDF; and if the side AB be not equal to the side DE, one of them must be the greater. Let AB be the greater of the two, and make BG equal to DE, and join CG; therefore, because BG is equal to DE, and BC to EF, the two sides GB, BC are equal to the two sides DE, EF,



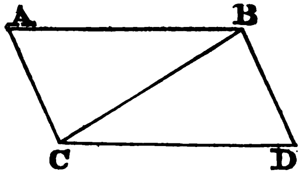
each to each; and the angle GBC is equal to the angle DEF; therefore the base CG is equal (4. 1.) to the base DF, and the triangle GBC is equivalent (4. 1.) to the triangle DEF; therefore the angle GCB is equal to the angle DFE: but DFE is, by the hypothesis, equal to the angle ACB; wherefore, also, the angle GCB is equal to the angle ACB, the less to the greater, which is impossible; therefore AB is not unequal to DE, that is, it is equal to it, and BC is equal to EF; therefore the two AB, BC are equal to the two DE, EF, each to each; and the angle ABC is equal to the angle DEF; the base, therefore, AC, is equal (4. 1.) to the base DF, and the triangle ABC is equivalent to the triangle DEF. Therefore, if two triangles have two angles of one equal to two angles of the other, each to each; and one side equal to one side, the triangles shall be equivalent. Which was to be proved.

PROP. VII. THEOR.

If two of the sides of a quadrangle are equal and parallel, the remaining sides are also equal and parallel.

Let ABDC be a quadrangle, and let the sides AB and CD be equal and parallel; AC and BD are also equal and parallel. Join CB.

Because AB is parallel to CD, the corresponding angles ABC and BCD are equal (3. 2.), and the side AB is equal to CD by the hy-



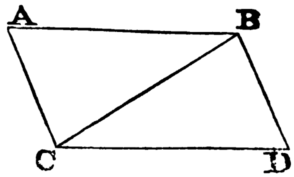
pothesis, and BC is common; therefore (4. 1.) the triangles ABC, BCD are equivalent, and the side AC is equal to the side BD, and the angle ACB to the angle CBD; and therefore (Cor. 1. 2. 2.) AC is parallel to BD.

PROP. VIII. THEOR.

If two of the sides of a quadrangle are equal to the opposite two, each to each, they are also parallel.

Let ABDC be a quadrangle, and let the side AB be equal to the side CD, and let AC be also equal to BD: AB shall be parallel to CD, and AC to BD. Join BC.

Then, because the two triangles ABC and BCD have the side AB



equal to the side CD, and AC to BD by the hypothesis, and the third side BC common (7. 1.), they are equivalent; and therefore the angle ABC is equal to the angle BCD; and therefore AB (Cor. 1. 2. 2.) is parallel to CD; and because the angle ACB is equal to the angle CBD, AC is parallel to BD. Q. E. D.

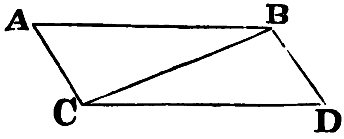
PROP. IX. THEOR.

The opposite sides and angles of a parallelogram are equal.

In the quadrangle ABDC, let AB be parallel to CD, and AC to BD. AB is equal to CD, and AC to BD. Join BC.

Because AB is parallel to CD, and BC meets them, the corresponding angles ABC, BCD are equal (3. 2.) to one another; and because AC is parallel to BD, and BC meets them, the corresponding angles ACB, CBD are equal (3. 2.) to one another; wherefore the two triangles ABC, CBD have two angles ABC, BCA in one, equal to two

angles BCD, CBD in the other, each to each, and one side BC, common to the two triangles; therefore the two triangles are equivalent (6. 2.), and their other sides are equal, each to each; and the third angle of the one is equal to the third angle of the other, viz. the side AB to the side CD, and AC to BD, and the angle BAC is equal to the angle BDC; and because the angle ABC is equal to the angle BCD, and the angle CBD to the angle ACB; therefore the whole angle ABD is equal to the whole angle ACD; and the angle BAC has been shown to be equal to the angle BDC. Therefore, &c.



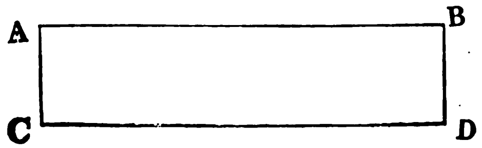
Cor. Because the triangles ABC, BCD are equivalent, as proved above; therefore the diagonal bisects the parallelogram.

PROP. X. THEOR.

Two straight lines which are at the same distance at each extremity, are parallel.

Let the straight lines AB and CD be at the same distance at A as at B; they shall be parallel.

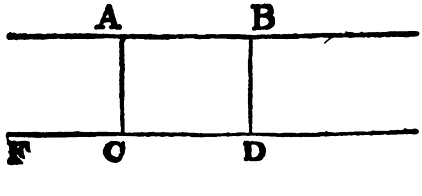
From A draw AC perpendicular to CD, and from B, BD also perpendicular to CD. Then in the quadrangle ABDC, the sides AC and BD are equal by the hypothesis, and parallel, because they are both perpendicular (2 Cor. 2. 2.) to CD; therefore (7. 2.) the remaining sides AB and CD, are also equal and parallel. Q. E. D.



PROP. XI. THEOR.

The perpendiculars between parallel lines are equal.

Let AB and FD be parallel lines, and AC and BD perpendiculars between them; AC is equal to BD.



Because the straight line FD falls on the straight lines AC, BD, making the corresponding angles ACF, BDC, equal (1 Cor. 2. 2.), being both right angles (2 Cor. 1. 1.), AC is parallel to BD, and ABDC is a parallelogram; and therefore AC is equal (9. 2.) to BD. Therefore the perpendiculars between parallel lines are equal. Which was to be proved.

Cor. If parallel lines are produced indefinitely from either extremity, they remain always at the same distance from each other.

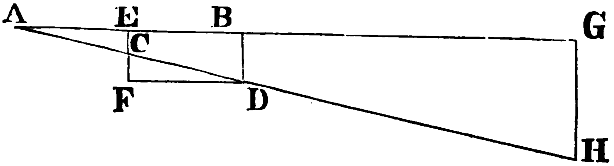
PROP. XII. THEOR.

If two lines make an angle, and two points be taken in one of the lines, so that one of the points is twice the distance of the other from the angular point, it shall also be twice the distance from the other line.

Let the straight lines AB and AD make an angle, and let the points C and D be taken in AD, so that AD shall be double of AC, and CE and BD be drawn perpendicular to EB; BD shall be double of CE.

Produce CE to F, make CF equal to CE, and join FD.

Because the two triangles ACE, FDC have the two sides AC, CE equal to the two sides CD, CF, each to each; and likewise have the contained angle ACE equal (3. 1.) to the contained angle FCD, being vertical angles, therefore the triangles are equivalent (4. 1.), and



the angle CDF is equal to the angle CAE; and because the straight line AD falls on the straight lines AB, DF, making the angle DAE or CAE, equal to the corresponding angle ADF or CDF; therefore, FD is parallel (1 Cor. 2. 2.) to AB or BE; and because EF and BD are both perpendicular to BE, the angles FEB, DBE are each right angles, and are together equal to two right angles; therefore EF, BD are parallel (2 Cor. 2. 2.); and therefore BEFD is a parallelogram, and BD is equal (9. 2.) to EF; and because CF was made equal to CE, EF is double of CE, and therefore BD is also double of CE. Therefore, if two lines make an angle, and two points be taken in one of the lines, so that one of the points is twice the distance of the other from the angular point, it shall also be twice the distance from the other line. Which was to be proved.

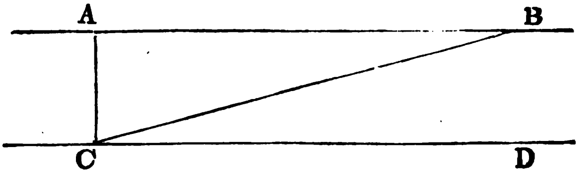
Cor. Produce AB and AD to G and H, so that AH shall be double of AD, and draw GH perpendicular to AG: it may be proved in the same manner, that GH is double of BD, and if they are continually produced, the perpendiculars will be continually increased; and therefore they may be produced until the perpendicular is greater than any given straight line.

PROP. XIII. THEOR.

If a straight line meet two straight lines, so as to make the two interior angles on the same side of it, taken together, less than two right angles, these straight lines being continually produced, shall at length meet upon that side on which are the angles, which are less than two right angles.

Let the straight line AC fall on the straight lines AB, BC, so as to make the angles BAC, ACB together less than two right angles; AB and CB, if continually produced, shall at length meet.

Through C draw CD, parallel to AB; then because AB is parallel to CD, the angles BAC, ACD are together equal (1 Cor. 3. 2.) to two right angles; but the angles BAC, ACB are, by the hypothesis, together less than two right angles; and are therefore less than the angles BAC, ACD; take away the common angle BAC, and the remaining angle ACB is less than the remaining angle ACD; and the angle ACD is composed of the angles ACB, BCD; and because the straight lines BC, CD make an angle, if they are produced, the distance between their extremities will be increased, and the more they are produced the more it will be increased (Cor. 12. 2.); and therefore they may be produced, till the distance between their extremities is equal to the distance between the parallels AB, CD (Cor. 12. 2.); and because parallel lines when produced, remain always at the same distance from each other (Cor. 11. 2.), BC will then meet AB. Therefore, if a straight line meet two straight lines, so as to make the two interior angles on the same side of it, taken together, less than two right angles, these straight lines being continually produced, shall at length meet upon that side on which are the angles, which are less than two right angles. Which was to be proved.

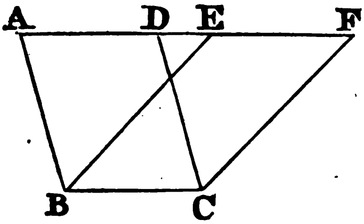


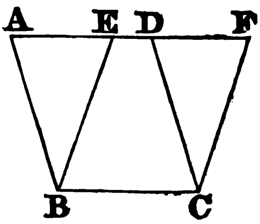
PROP. XIV. THEOR.

Parallelograms of equal base and altitude, are equal.

Let the parallelograms ABCD, EBCF, be of equal base and altitude, they shall also be equal. Let the parallelograms be applied, so that the bases coincide with BC: and join AF.

Then, because the altitude of the parallelogram ABCD, is equal to





the altitude of the parallelogram EBCF, by the hypothesis, the perpendicular from F on BC, shall be equal to the perpendicular from A on BC (Def. 16. 2.); therefore the straight line AF shall be parallel to the straight line BC (10. 2.). But the straight line AD, passing through the point A, is also parallel to BC; and therefore AD coincides with AF: for the same reason EF coincides with AF.

And because ABCD is a parallelogram, AD is equal to BC (9. 2.). For the same reason, EF is equal to BC, and AD being equal to EF, and DE common, the whole or the remainder AE is equal to the whole or the remainder DF. AB is also equal to AC, and the two EB, AB, are equal to the two, FD, DC, each to each; and the angle FDC is equal to the angle EAB, because they are corresponding angles (3. 2.); therefore the triangle EAB is equivalent to the triangle FDC (4. 1.). From the quadrangle ABCF, take the triangle FDC, and from the same quadrangle take the triangle EAB, and the remainder, the parallelogram ABCD, will be equal to the remainder, the parallelogram EBCF. Q. E. D.

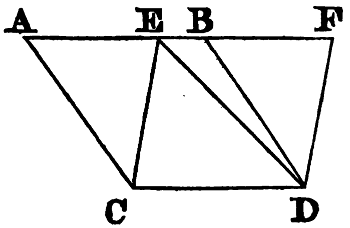
Cor. Equal parallelograms of equal base, are of equal altitude.

PROP. XV. THEOR.

A triangle is equal to half a parallelogram, of equal base and altitude.

Let the triangle ECD, and the parallelogram ABDC, be of equal base and altitude: ECD shall be equal to half ABCD. Let them be applied so as to coincide with the base CD, and complete the parallelogram ECDF.

Because the parallelograms ABDC and ECDF, are on the same



base and of the same altitude, they are equal (14. 2.); but the triangle ECD is half the parallelogram ECDF; and therefore it is equal to half the parallelogram ABCD. Q. E. D.

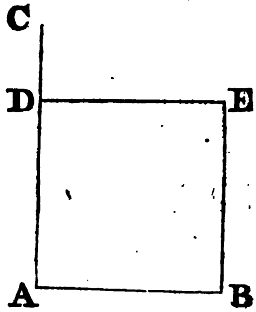
COR. I. Triangles of equal base and altitude, are equal.
COR. II. Equal triangles of equal bases, are of equal altitude.

PROP. XVI. PROB.

To describe a square upon a given straight line.

Let AB be the given straight line; it is required to describe a square upon AB.

From the point A, draw AC at right angles (9. 1.) to AB, and make AD equal to AB; and through the point D, draw DE parallel (4. 2.) to AB, and through B draw BE parallel (4. 2.) to AD; therefore ABED is a parallelogram; whence AB is equal (9. 2.) to DE, and AD to BE; but AB is equal to AD; therefore the four straight lines AB, AD, BE, DE, are equal to one another, and the parallelogram ABED is equilateral; likewise all its angles are right angles; because the straight line AD, meeting the parallels AB, DE, the angles BAD, ADE are equal to two right angles (1 Cor. 3. 2.); but BAD is a right angle, therefore also, ADE is a right angle; but the opposite angles of parallelograms are equal (9. 2.); therefore each of the opposite angles ABE, BED is a right angle; wherefore the figure ABED is rectangular, and it has been demonstrated that it is equilateral; it is therefore a square (Def. 5. 2.), and it is described upon the given straight line AB. Which was to be done.



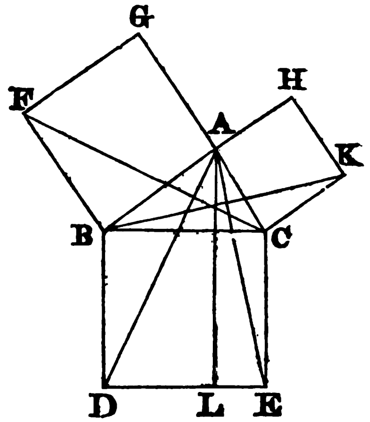
COR. Hence every parallelogram that has one right angle, has all its angles right angles.

PROP XVII. THEOR.

In any right angled triangle, the square which is described upon the side subtending the right angle, is equal to the squares described upon the sides which contain the right angle.

Let ABC be a right angled triangle, having the right angle BAC; the square described upon the side BC, is equal to the squares described upon AB, AC.

On BC describe (16. 2.) the square BDEC, and on AB, AC, the squares BG, CH; and through A draw AL, parallel (4. 2.) to BD or CE, and join AD, CF; then, because each of the angles BAC, BAG, is a right angle (Def. 5. 2.), the two straight lines AC, AG, upon the



opposite sides of AB, make with it at the point A, the adjacent angles equal to two right angles; therefore AC is in the same straight line (2. 1.) with AG: for the same reason, AB and AH are in the same straight line; and because the angle DBC is equal to the angle FBA, each of them being a right angle (Def. 5. 2.), add to each the angle ABC, and the whole angle DBA is equal to the whole FBC; and because the two sides AB, BD are equal to the two BF, BC, each to each, and the angle DBA equal to the angle FBC; therefore the base AD is equal (4. 1.) to the base CF, and the triangle ABD is equivalent to the triangle FBC; now the parallelogram BL is double (15. 2.) of the triangle ABD, because they are upon the same base BD, and between the same parallels BD, AL; and the square BG is double (15. 2.) of the triangle FBC, because these also are upon the same base BF, and between the same parallels BF, CG. But the doubles of equals are equal to one another; therefore the parallelogram BL, is equal to the square BG; and in the same manner, by joining AE, BK, it is demonstrated that the parallelogram CL is equal to the square CH; therefore the whole square BDEC is equal to the two squares BG, CH; and the square BDEC is described upon the straight line BC, and the squares BG, CH upon AB, AC; therefore the square upon the side BC, is equal to the squares upon the sides AB, AC. Therefore, in any right angled triangle, the square which is described upon the side subtending the right angle, is equal to the squares described upon the sides which contain the right angle. Which was to be proved.

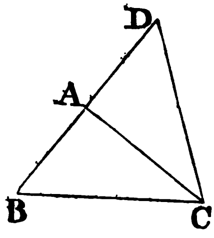
PROP. XVIII. THEOR.

If the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it, the angle contained by these two sides is a right angle.

If the square described upon BC, one of the sides of the triangle

ABC, be equal to the squares upon the other sides AB, AC, the angle BAC is a right angle.

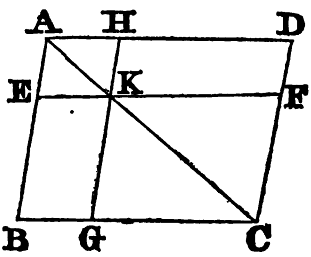
From the point A, draw AD at right angles (9. 1.) to AC, and make AD equal to AB, and join CD; then, because AD is equal to AB, the square of AD is equal to the square of AB; to each of these add the square of AC; therefore the squares of AD, AC are equal to the squares of AB, AC; but the square of CD is equal (17. 2.) to the squares of AD, AC, because DAC is a right angle; and the square of BC, by hypothesis, is equal to the squares of AB, AC; therefore the square of DC, is equal to the square of BC: and therefore, also, the side DC is equal to the side BC. And because the side AD is equal to AB, and AC common to the two triangles DAC, BAC, the two sides AD, AC are equal to the two sides AB, AC, and the base DC is equal to the base BC; therefore the angle DAC is equal (7. 1.) to the angle BAC; but DAC is a right angle; therefore also BAC is a right angle. Therefore, If the square, &c.



PROP. XIX. THEOR.

The complements of the parallelograms which are about the diagonal of any parallelogram, are equal to one another.

Let ABCD be a parallelogram, of which the diagonal is AC, and EH, FG the parallelograms about AC, that is, through which AC passes, and BH, DK, the other parallelograms which make up the whole figure ABCD, which are therefore called the complements; the complement BK is equal to the complement DK.



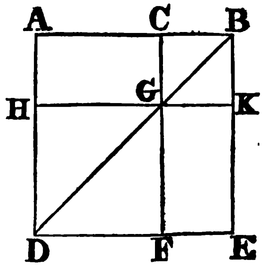
Because ABCD is a parallelogram, and AC its diagonal, the triangle ABC is equal (Cor. 9. 2.) to the triangle ADC; and because AEKH is a parallelogram, the diagonal of which is AK, the triangle AEK is equal to the triangle AHK; for the same reason, the triangle KGC is equal to the triangle KFC; then, because the triangle AEK is equal to the triangle AHK, and the triangle KGC equal to the triangle KFC; the triangle AEK, together with the triangle KGC, is equal to the triangle AHK, together with the triangle KFC; but the whole triangle ABC, is equal to the whole triangle ADC; therefore the remaining complement BK, is equal to the remaining complement DK. Therefore, the complements of the parallelograms which are about the diagonal of any parallelogram, are equal to one another. Which was to be proved.

PROP. XX. THEOR.

If a straight line be divided into any two parts, the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the parts.

Let the straight line AB be divided into any two parts in C; the square of AB is equal to the squares of AC, CB, and to twice the rectangle contained by AC, CB.

Upon AB describe (16. 2.) the square ADEB, and join BD, and through C draw (4. 2.) CGF parallel to AD or BE, and through G draw HK parallel to AB or DE; and because CF is parallel to AD, and BD falls upon them, the angle BGC is equal (3. 2.) to the corresponding angle ADB; but ADB is equal (5. 1.) to the angle ABD, because BA is equal to AD, being sides of a square; wherefore the angle CGB is equal to the angle GBC, and therefore the side BC is equal (6. 1.) to the side CG; but CB is equal also (9. 2.) to GK, and CG to BK; wherefore the figure CGKB is equilateral: it is likewise rectangular; for CG is parallel to BK, and CB meets them, the angles KBC, GCB are therefore equal to two right angles; and KBC is a right angle, wherefore GCB is a right angle; and therefore also the angles CGK, GKB opposite to these are right angles, and CGKB is rectangular; but it is also equilateral, as was demonstrated; wherefore it is a square, and it is upon the side CB: for the same reason HF also is a square, and it is upon the side HG which is equal to AC; therefore HF, CK are the squares of AC, CB: and because the complement AG is equal (19. 2.) to the complement GE, and that AG is the rectangle contained by AC, CB, for GC is equal to CB; therefore GE is also equal to the rectangle AC, CB; wherefore AG, GE are equal to twice the rectangle AC, CB; and HF, CK are the squares of AC, CB; wherefore the four figures HF, CK, AG, GE are equal to the squares of AC, CB, and to twice the rectangle AC, CB; but HF, CK, AG, GE make up the whole figure ADEB which is the square of AB; therefore the square of AB is equal to the squares of AC, CB, and twice the rectangle AC, CB. Wherefore, If a straight line, &c. Q. E. D.



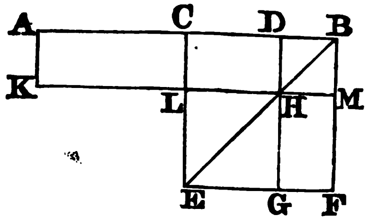
COR. The parallelograms about the diagonal of a square are likewise squares.

PROP. XXI. THEOR.

If a straight line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equal to the square of half the line.

Let the straight line AB be divided into two equal parts in the point C, and into two unequal parts at the point D; the rectangle AD, DB, together with the square of CD, is equal to the square of CB.

Upon CB describe (16. 2.) the square CEFB, join BE, and through D draw (4. 2.) DHG parallel to CE or BF; and through H draw KLM parallel to CB or EF; and also through A draw AK parallel to CL or BM; and because the complement CH is equal (19. 2.) to the complement HF, to each of these add DM, therefore the whole CM is equal to the whole DF; but CM is equal (14. 2.) to AL, because AC is equal to CB; therefore also AL is equal to DF; to each of these add CH, and the whole AH is equal to DF and CH; but AH is the rectangle contained by AD, DB, for DH is equal (Cor. 20. 2.) to DB; and DF together with CH is the gnomon CMG; therefore the gnomon CMG is equal to the rectangle AD, DB; to each of these add LG, which is equal to the square of CD, therefore the gnomon CMG together with LG is equal to the rectangle AD, DB, together with the square of CD; but the gnomon CMG and LG make up the whole figure CEFB, which is the square of CB; therefore the rectangle AD, DB, together with the square of CD, is equal to the square of CB. Wherefore, If a straight line, &c. Q. E. D.

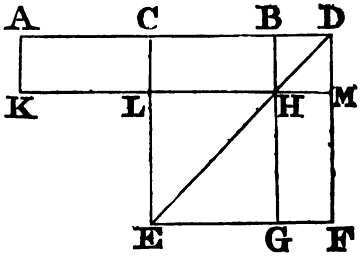


PROP. XXII. THEOR.

If a straight line be bisected, and produced to any point; the rectangle contained by the whole line thus produced, and the part of it produced, together with the square of half the line bisected, is equal to the square of the straight line which is made up of the half and the part produced.

Let the straight line AB be bisected in C, and produced to the point D; the rectangle AD, DB, together with the square of CB, is equal to the square of CD.

Upon CD describe (16. 2.) the square CEFD, join DE, and through B draw (4. 2.) BHG parallel to CE or DF, and through H draw KLM parallel to AD or EF, and also through A draw AK parallel to CL or DM: and because AC is equal to CB, the rectangle AL is equal



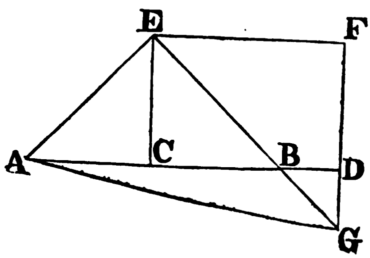
(14. 2.) to CH; but CH is equal (19. 2.) to HF; therefore also AL is equal to HF: to each of these add CM; therefore the whole AM is equal to the gnomon CMG: and AM is the rectangle contained by AD, BD, for DM is equal (Cor. 20. 2.) to DB: therefore the gnomon CMG is equal to the rectangle AD, DB; add to each of these LG, which is equal to the square of CB; therefore the rectangle AD DB, together with the square of CB, is equal to the gnomon CMG and the figure LG: but the gnomon CMG and LG make up the whole figure CEFD, which is the square of CD; therefore the rectangle AD, DB, together with the square of CB, is equal to the square of CD. Wherefore, if a straight line, &c. Q. E. D.

PROP. XXIII. THEOR.

If a straight line be bisected, and produced to any point, the square of the whole line thus produced, and the square of the part of it produced, are together double of the square of half the line bisected, and of the square of the line made up of the half and the part produced.

Let the straight line AB be bisected in C, and produced to the point D; the squares of AD, DB are double of the squares of AC, CD.

From the point C draw (9. 1.) CE at right angles to AB, and make it equal to AC or CB, and join AE, EB; through E draw (4. 2.) EF parallel to AB, and through D draw DF parallel to CE; and because the straight line EF meets the parallels EC, FD, the angles CEF, EFD are equal (1 Cor. 3. 2.) to two right angles; and therefore the angles BEF, EFD are less than two right angles: but straight lines which with another straight line make the interior angles upon the same side less than two right angles, do meet (13. 2.) if produced far enough; therefore EB, FD shall meet, if produced, toward BD; let them meet in G, and join AG; then because AC is equal to CE, the angle CEA is equal (5. 1.) to the angle EAC; and the angle ACE is



a right angle; therefore each of the angles CEA, EAC is half a right angle: for the same reason, each of the angles CEB, EBC is half a right angle; therefore AEB is a right angle; and because EBC is half a right angle, DBG is also (3. 1.) half a right angle, for they are vertically opposite; but BDG is a right angle, because it is equal to the corresponding angle DCE; therefore the remaining angle DGB is half a right angle, and is therefore equal to the angle DBG; wherefore also the side BD is equal (6. 1.) to the side DG. Again, because EGF is half a right angle, and that the angle at F is a right angle, because it is equal (9. 2.) to the opposite angle ECD, the remaining angle FEG is half a right angle, and equal to the angle EGF; wherefore also the side GF is equal (6. 1.) to the side FE. And because EC is equal to CA, the square of EC is equal to the square of CA; therefore the squares of EC, CA are double of the square of CA: but the square of EA is equal (17. 2.) to the squares of EC, CA; therefore the square of EA is double of the square of AC: again, because GF is equal to FE, the square of GF is equal to the square of FE; and therefore the squares of GF, FE are double of the square of EF; but the square of EG is equal (17. 2.) to the squares of GF, FE; therefore the square of EG is double of the square of EF; and EF is equal to CD; wherefore the square of EG is double of the square of CD; but it was demonstrated that the square of EA is double of the square of AC; therefore the squares of AE, EG are double of the squares of AC, CD; and the square of AG is equal (17. 2.) to the squares of AE, EG; therefore the square of AG is double of the squares of AC, CD: but the squares of AD, DG are equal (17. 2.) to the square of AG; therefore the squares of AD, DG are double of the squares of AC, CD: but DG is equal to DB; therefore the squares of AD, DB are double of the squares of AC, CD. Wherefore, If a straight line, &c. Q. E. D.

PROP. XXIV. PROB.

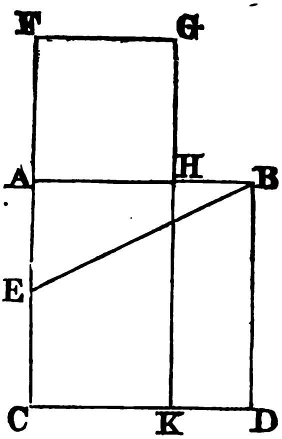
To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square of the other part.

Let AB be the given straight line: it is required to divide it into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square of the other part.

Upon AB describe (16. 2.) the square ABDC; bisect (8. 1.) AC in E, and join BE; produce CA to F, and make EF equal to EB; and upon AF describe (16. 2.) the square FGHA; AB is divided in H, so that the rectangle AB, BH is equal to the square of AH.

Produce GH to K; because the straight line AC is bisected in E, and produced to the point F, the rectangle CF, FA, together with the square of AE, is equal (22. 2.) to the square of EF: but EF is equal to EB; therefore the rectangle CF, FA, together with the square of

AE, is equal to the square of EB: and the squares of BA, AE are equal (17. 2.) to the square of EB, because the angle EAB is a right angle; therefore the rectangle CF, FA, together with the square of AE, is equal to the squares of BA, AE: take away the square of AE, which is common to both, therefore the remaining rectangle CF, FA is equal to the square of AB; and the figure FK is the rectangle contained by CF, FA, for AF is equal to FG; and AD is the square of AB; therefore FK is equal to AD: take away the common part AK, and the remainder FH is equal to the remainder HD, and HD is the rectangle contained by AB, BH, for AB is equal to BD; and FH is the square of AH: therefore the rectangle AB, BH is equal to the square of AH: wherefore the straight line AB is divided in H, so that the rectangle AB, BH is equal to the square of AH. Which was to be done.

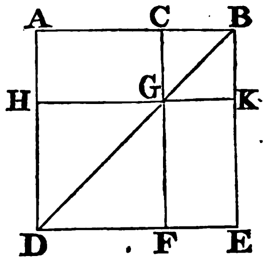


PROP. XXV. THEOR.

If a straight line be divided into any two parts, the squares of the whole line, and of one of the parts are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.

Let the straight line AB be divided into any two parts in the point C; the squares of AB, BC are equal to twice the rectangle AB, BC, together with the square of AC.

Upon AB describe (16. 2.) the square ADEB, and construct the figure as in the preceding propositions: and because AG is equal (19. 2.) to GE, add to each of them CK; the whole AK is therefore equal to the whole CE; therefore AK, CE are double of AK: but AK, CE are the gnomon AKF, together with the square CK; therefore the gnomon AKF, together with the square CK, is double of KA: but twice the rectangle AB, BC is double of AK, for BK is equal (Cor. 20. 2.) to BC: therefore the gnomon AKF, together with the square CK, is equal to twice the rectangle AB, BC: to each of these equals add HF, which is equal to the square of AC: therefore the gnomon AKF, together with the squares CK, HF, is equal to twice the rectangle AB, BC, and the square of AC: but the gnomon AKF, together with the



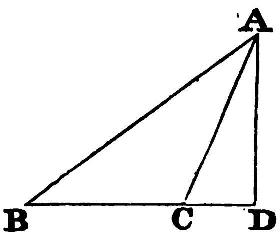
squares CK, HF, make up the whole figure ADEB and CK, which are the squares of AB and BC: therefore the squares of AB and BC are equal to twice the rectangle AB, BC, together with the square AC. Wherefore, if a straight line, &c. Q. E. D.

PROP. XXVI. THEOR.

In obtuse angled triangles, if a perpendicular be drawn from any of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle, is greater than the squares of the sides containing the obtuse angle, by twice the rectangle contained by the side upon which when produced the perpendicular falls, and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.

Let ABC be an obtuse angled triangle, having the obtuse angle ACB, and from the point A let AD be drawn (10. 1.) perpendicular to BC produced; the square of AB is greater than the squares of AC, CB by twice the rectangle BC, CD.

Because the straight line BD is divided into two parts in the point C, the square of BD is equal (20. 2.) to the squares of BC, CD, and twice the rectangle BC, CD: to each of these equals add the square of DA; and the squares of BD, DA are equal to the squares of BC, CD, DA, and twice the rectangle BC, CD: but the square of BA is equal (17. 2.) to the squares of BD, DA, because the angle at D is a right angle; and the square of CA is equal to the squares of CD, DA: therefore the square of BA is equal to the squares of BC, CA, and twice the rectangle BC, CD; that is, the square of BA is greater than the squares of BC, CA, by twice the rectangle BC, CD. Therefore, in obtuse angled triangles, &c. Q. E. D.

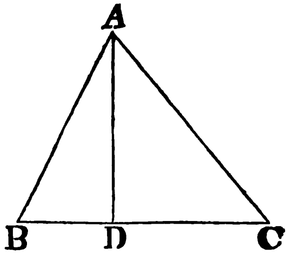


PROP. XXVIII. THEOR.

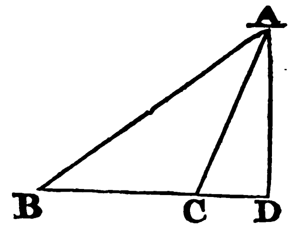
In every triangle, the square of the side subtending any of the acute angles is less than the squares of the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall upon it from the opposite angle, and the acute angle.

Let ABC be any triangle, and the angle at B one of its acute angles, and upon BC, one of the sides containing it, let fall the perpendicular (10. 1.) AD from the opposite angle: the square of AC, opposite to the angle B, is less than the squares of CB, BA, by twice the rectangle CB, BD.

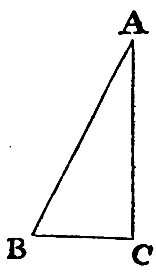
First, Let AD fall within the triangle ABC; and because the straight line CD is divided into two parts in the point D, the squares of CB, BD are equal (25. 2.) to twice the rectangle contained by CB, BD, and the square of DC: to each of these equals add the square of AD; therefore the squares of CB, BD, DA are equal to twice the rectangle CB BD, and the squares of AD, DC: but the square of AB is equal (17. 2.) to the squares of BD, DA, because the angle BDA is a right angle, and the square of AC is equal to the squares of AD, DC: therefore the squares of CB, BA are equal to the square of AC, and twice the rectangle CB, BD, that is, the square of AC alone is less than the squares of CB, BA by twice the rectangle CB, BD.



Secondly, Let AD fall without the triangle ABC: then, because the angle at D is a right angle, the angle ACB is greater (5. 2.) than a right angle; and therefore the square of AB is equal (26. 2.) to the squares of AC, CB, and twice the rectangle BC, CD: to these equals add the square of BC, and the squares of AB, BC are equal to the square of AC, and twice the square of BC, and twice the rectangle BC, CD: but because BD is divided into two parts in C, the rectangle DB, BC is composed of the rectangle BC, CD and the square of BC: and the doubles of these are equal: therefore the squares of AB, BC are equal to the square of AC, and twice the rectangle DB, BC: therefore the square of AC alone is less than the squares of AB, BC by twice the rectangle DB, BC.



Lastly, Let the side AC be perpendicular to BC; then is BC the straight line between the perpendicular and the acute angle at B; and it is manifest that the squares of AB, BC are equal (17. 2.) to the square of AC and twice the square of BC. Therefore, in every triangle, &c. Q. E. D.



THE

ELEMENTS

OF

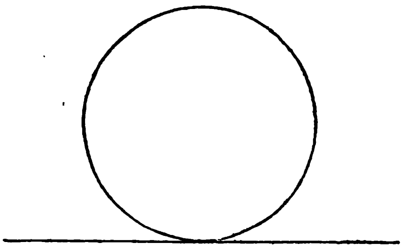
GEOMETRY.

BOOK III.

DEFINITIONS.

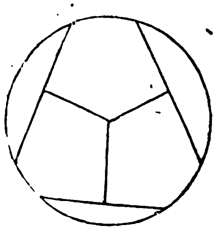
I.

When a straight line meets a circle, and being produced does not cut the circle, it may be said to touch the circle, and may be called a tangent.



II.

When the perpendiculars drawn to straight lines from the centre of a circle are equal, the straight lines may be said to be equally distant from the centre.



III.

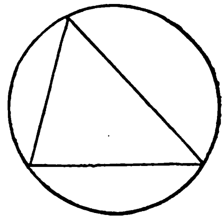
And the straight line on which the greater perpendicular falls, may be said to be farther from the centre.

IV.

The part of a circle contained by a straight line and the circumference which it cuts off, may be called a segment of a circle, and the straight line may be called the base of the segment.

V.

The angle contained by two straight lines drawn from any point in the circumference of the segment, to the extremities of the base of the segment, may be called the angle in a segment.



VI.

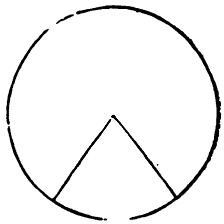
An angle may be said to stand on the circumference intercepted between the straight lines which contain the angle.

VII.

When straight lines are said to be drawn from the centre, it is to be understood that they are drawn to the circumference.

VIII.

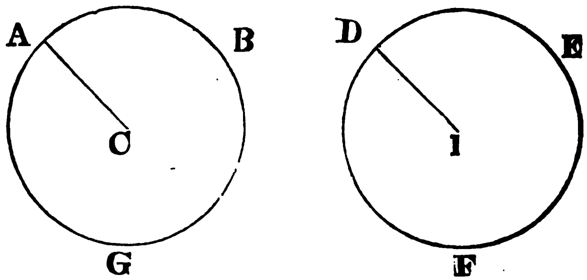
The figure contained by two straight lines drawn from the centre, and the circumference between them, may be called a sector of a circle.



PROPOSITION I. THEOREM.

If the radius of one circle is equal to the radius of another circle, the circles are also equal.

Let ABG, DEF be two circles, having the radius AC of the one circle equal to the radius DI of the other circle, the circles will be equal. For, if the circle ABG be applied to the circle DEF, so that the centre C of the one, shall be upon the centre I of the other; then,

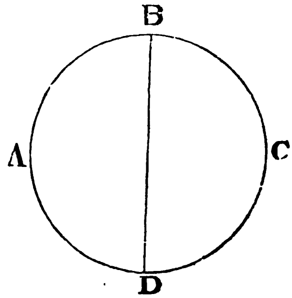


because AC is equal to DI, and because all the straight lines from the centre to the circumference are equal (Def. 54. 1.), the circumference of the circle ABG will fall upon the circumference of the circle DEF, and the circle ABG will be equal to the circle DEF. Therefore, if the radius of one circle is equal to the radius of another circle, the circles are also equal. Which was to be proved.

PROP. II. THEOR.

The diameter divides the circle into two equivalent parts.

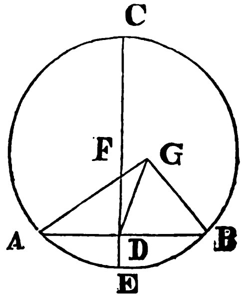
Let ABCD be a circle, and BD the diameter; DAB is equivalent to BCD. For, if DAB be applied to BCD, because all the lines from the centre to the circumference are equal (Def. 54. 1.), the circumference DAB will fall upon the circumference BCD, and the segment DAB will be equivalent to the segment BCD. Therefore, the diameter divides the circle into two equivalent parts. Which was to be proved.



PROP. III. PROB.

To find the centre of a given circle.

Let ABC be the given circle; it is required to find its centre. Draw any straight line AB in the circle ABC, and bisect (8. 1.) it in D; from the point D draw (9. 1.) DC at right angles to AB, and produce it to E, and bisect (8. 1.) CE in F; the point F is the centre of the circle ABC. For, if it be not, let, if possible, G be the centre, and join GA, GD, GB; then, because DA is equal to DB, and DG common to the two triangles ADG, BDG, the two sides AD, DG are equal to the two sides BD, DG, each to each; and the base AG is equal to the base BG, because they are drawn from the centre to the circumference (Def. 54. 1.); therefore the angle ADG is equal (7. 1.) to the angle BDG; but when a straight line standing upon another straight line, makes the adjacent angles equal to one another, each of the angles is a right angle (Def. 27. 1.); therefore BDG is a right angle; but BDF is likewise a right angle; wherefore the angle BDF is equal to the angle BDG, the greater to the less, which is impossible; therefore G is not the centre of the circle ABC. In the same manner it can be shewn, that no other point but F is the centre; that is, F is the centre of the circle ABC. Which was to be found.



Cor. If a straight line bisect another straight line in the circle at right angles, the centre of the circle is in the line which bisects the other.

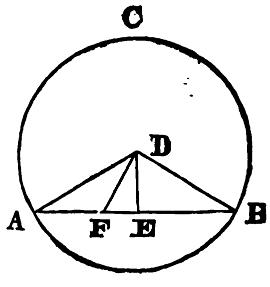
PROP. IV. THEOR.

If two points be taken in the circumference of a circle, the straight line which joins them shall fall wholly within the circle.

Let ABC be a circle, A and B any points in the circumference; the straight line AB is wholly within the circle.

Find the centre of the circle D (3. 3.), draw DC perpendicular to AB (10. 1.), and join DA, DB; and draw DF any straight line from D upon AB; DF is less than DA.

Because DF is nearer the perpendicular DE, it is less than DA (13. 1.): in the same way it may be shewn that any other straight line from D to AB is less than DA; and therefore DF to meet the circumference, must



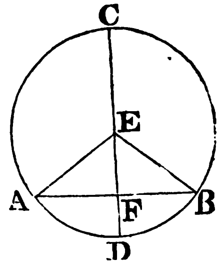
be produced beyond AB. Therefore the circumference is beyond AB, or AB is wholly within the circle. Q. E. D.

PROP. V. THEOR.

If a straight line drawn through the centre of a circle, bisect a straight line in it which does not pass through the centre, it shall cut it at right angles; and if it cuts it at right angles, it shall bisect it.

Let ABC be a circle; and let CD, a straight line drawn through the centre, bisect any straight line AB, which does not pass through the centre, in the point F; it cuts it also at right angles.

Take (3. 3.) E the centre of the circle, and join EA, EB; then because AF is equal to FB, and FE common to the two triangles AFE, BFE, there are two sides in the one equal to two sides in the other; and the base EA is equal to the base EB; therefore the angle AFE is equal (7. 1.) to the angle BFE; but when a straight line standing upon another, makes the adjacent angles equal to one another, each of them is a right (Def. 27. 1.) angle; therefore each of the angles AFE, BFE is a right angle; wherefore the straight line CD drawn through the centre bisecting another AB that does not pass through the centre, cuts the same at right angles.



But let CD cut AB at right angles; CD also bisects it, that is, AF is equal to FB.

The same construction being made, because EA, EB from the centre are equal to one another, the angle EAF is equal (5. 1.) to the angle EBF; and the right angle AFE is equal to the right angle BFE; therefore in the two triangles EAF, EBF there are two angles in one equal to two angles in the other, and the side EF which is opposite to one of the equal angles in each, is common to both; therefore the other sides are equal (6. 2.); AF therefore is equal to FB. Wherefore, if a straight line, &c. Q. E. D.

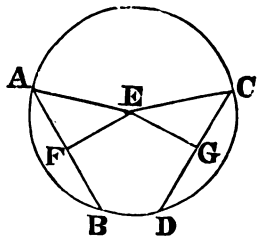
PROP. VI. THEOR.

Equal straight lines in a circle are equally distant from the centre; and those which are equally distant from the centre, are equal to one another.

Let the straight lines AB, CD in the circle ABDC be equal to one another; they are equally distant from the centre.

Take E the centre of the circle ABDC, and from it draw EF, EG perpendiculars to AB, CD; then, because the straight line EF passing through the centre cuts the straight line AB, which does not pass through the centre, at right angles, it also bisects (5. 3.) it; where-

fore AF is equal to FB, and AB double of AF; for the same reason CD is double of CG, and AB is equal to CD, therefore AF is equal to CG; and because AE is equal to EC, the square of AE is equal to the square of EC; but the squares of AF, FE are equal (17. 2.) to the square of AE, because the angle AFE is a right angle; and for the like reason the squares of EG, GC are equal to the square of EC; therefore the squares of AF, FE are equal to the squares of CG, GE, of which the square of AF is equal to the square of CG, because AF is equal to CG; therefore the remaining square of FE is equal to the remaining square of EG, and the straight line FE is therefore equal to EG; but straight lines in a circle are said to be equally distant from the centre, when the perpendiculars drawn to them from the centre are equal (Def. 2. 3.); therefore AB, CD are equally distant from the centre.



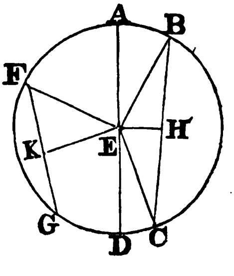
Next, if the straight lines AB, CD be equally distant from the centre, that is, if FE be equal to EG; AB is equal to CD; for, the same construction being made, it may, as before, be demonstrated that AB is double of AF, and CD, double of CG, and that the squares of EF, FA are equal to the squares of EG, GC; of which the square of FE is equal to the square of EG, because FE is equal to EG; therefore the remaining square of AF is equal to the remaining square of CG; and the straight line AF is therefore equal to CG, and AB is double of AF, and CD double of CG; wherefore AB is equal to CD. Therefore, equal straight lines, &c. Q. E. D.

PROP. VII. THEOR.

The diameter is the greatest straight line in a circle; and, of all others, that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the less.

Let ABCD be a circle, of which the diameter is AD, and the centre E; and let BC be nearer to the centre than FG; AD is greater than any straight line BC which is not a diameter, and BC greater than FG.

From the centre draw EH, EK perpendicular to BC, FG, and join EB, EC, EF; and because AE is equal to EB, and ED to EC, AD is equal to EB, EC; but EB, EC are greater (Def. 16. 1.) than BC; wherefore, also, AD is greater than BC.



And, because BC is nearer to the centre than FG, EH is less (Def. 3. 3.) than EK; but, as it was demonstrated in the preceding, BC is double of BH, and FG double of FK, and the squares of EH, HB are equal

to the squares of EK, KF, of which the square of EH is less than the square of EK, because EH is less than EK; therefore the square of BH is greater than the square of FK, and the straight line BH greater than FK; and therefore BC is greater than FG.

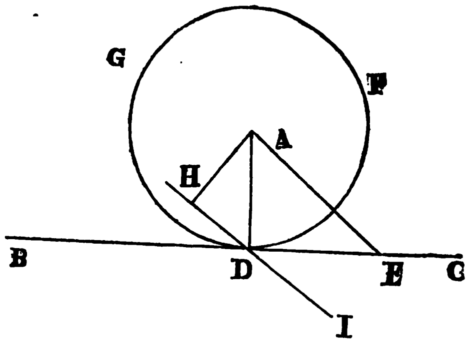
Next, Let BC be greater than FG; BC is nearer to the centre than FG, that is, the same construction being made, EH is less than EK: because BC is greater then FG, BH is likewise greater than KF: and the squares of BH, HE are equal to the squares of FK, KE, of which the square of BH is greater than the square of FK, because BH is greater than FK; therefore the square of EH is less than the square of EK, and the straight line EH less than EK. Wherefore the diameter, &c. Q. E. D.

PROP. VIII. THEOR.

A perpendicular drawn from the extremity of a radius, is a tangent to the circumference of a circle.

Let DFG be a circle, A the centre, AD a radius, and BC perpendicular to AD at the point D; BC is a tangent to the circumference DFG. Take E any point in BC, and join AE.

Because AD is perpendicular to BC, AE is oblique to BC, and therefore AE is longer than AD (13. 1.); and therefore the point E is without the circumference. It may be shewn in the same way, that



any point other than D, is without the circumference; and therefore BC does not cut the circumference; and therefore it is a tangent (Def. 1. 3.). Q. E. D.

Cor. Any other straight line through D, as HI, will cut the circumference.

Because AD is perpendicular to BC, and HI does not coincide with BC, AD is oblique to HI. Draw AH perpendicular to HI (10. 1); and because AH is perpendicular, and AD oblique to HI, AD is longer

than AH (12. 1.); and therefore the point H falls within the circumference; and therefore the straight line HI cuts the circumference.

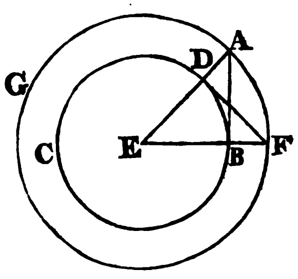
It may be shewn in the same way, that any other straight line through D, except BC, will cut the circumference. Q. E. D.

PROP. IX. THEOR.

To draw a straight line from a given point, either without or in the circumference, which shall touch a given circle.

First, Let A be a given point without the given circle BCD; it is required to draw a straight line from A which shall touch the circle.

Find (3. 3.) the centre E of the circle, and join AE; and from the centre E, at the distance AE, describe the circle AFG; from the point D draw (9. 1.) DF at right angles to EA, and join EBF, AB; AB touches the circle BCD.



Because E is the centre of the circles BCD, AFG, EA is equal to EF, and ED to EB; therefore the two sides AE, EB are equal to the two FE, ED, and they contain the angle at E common to the two triangles AEB, FED; therefore the base DF is equal to the base AB, and the triangle FED to the triangle AEB, and the other angles to the other angles (4. 1.); therefore the angle EBA is equal to the angle EDF; but EDF is a right angle, wherefore EAB is a right angle, and EB is drawn from the centre; but a straight line drawn from the extremity of a diameter, at right angles to it, touches the circle (8. 3.); therefore AB touches the circle; and it is drawn from the given point A. Which was to be done.

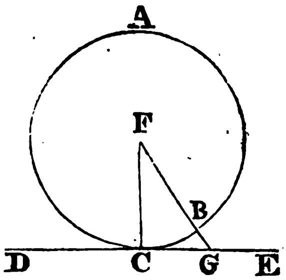
But if the given point be in the circumference of the circle, as the point D, draw DE to the centre E, and DF at right angles to DE; DF touches the circle.

PROP. X. THEOR.

If a straight line touch a circle, the straight line drawn from the centre to the point of contact, shall be perpendicular to the line touching the circle.

Let the straight line DE touch the circle ABC in the point C, take the centre F, and draw the straight line FC; FC is perpendicular to DE.

For if it be not, from the point F draw FBG perpendicular to DE (13. 1.); then the oblique line FC is longer than the perpendicular FG (13. 1.); therefore FC is greater than FG; but FC is equal to FB; therefore FB is greater than FG, the less than the greater, which is impossible; wherefore FG is not perpendicular to DE: in the same manner it may be shown, that no other is perpendicular to it besides FC, that is, FC is perpendicular to DE. Therefore, if a straight line, &c. Q. E. D.

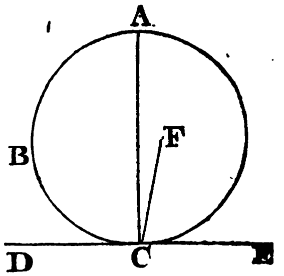


PROP. XI. THEOR.

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

Let the straight line DE touch the circle ABC in C, and from C let CA be drawn at right angles to DE; the centre of the circle is in CA.

For, if not, let F be the centre if possible, and join CF: because DE touches the circle ABC, and FC is drawn from the centre to the point of contact, FC is perpendicular (10. 3.) to DE; therefore FCE is a right angle; but ACE is also a right angle; therefore the angle FCE is equal to the angle ACE, the less to the greater, which is impossible: wherefore F is not the centre of the circle ABC; in the same manner it may be shown, that no other point which is not in CA, is the centre; that is, the centre is in CA. Therefore, if a straight line, &c. Q. E. D.



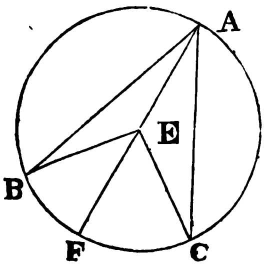
PROP. XII. THEOR.

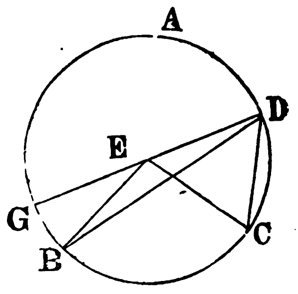
The angle at the centre of a circle is double of the angle at the circumference, upon the same base, that is, upon the same part of the circumference.

Let ABC be a circle, and BEC an angle at the centre, and BAC an angle at the circumference, which have the same circumference BC for their base; the angle BEC is double of the angle BAC.

First, Let E the centre of the circle be within the angle BAC, and

join AE, and produce it to F. Because EA is equal to EB, the angle EAB is equal (5. 1.) to the angle EBA; therefore the angles EAB, EBA are double of the angle EAB; but the angle BEF is equal (5. 2.) to the angles EAB, EBA; therefore also the angle BEF is double of the angle EAB; for the same reason, the angle FEC is double of the angle EAC; therefore the whole angle BEC is double of the whole angle BAC.

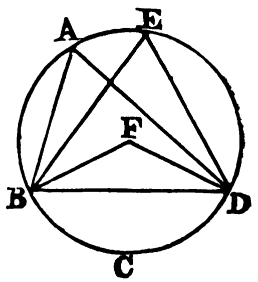




Again, Let BDC be inflectedt o the circumference, so that E the centre of the circle be without the angle BDC, and join DE, and produce it to C. It may be demonstrated, as in the first case, that the angle GEC is double of the angle GDC, and that GEB a part of the first, is double of GDB a part of the other; therefore the remaining angle BEC is double of the remaining angle BDC. Therefore, the angle at the centre, &c. Q. E. D.

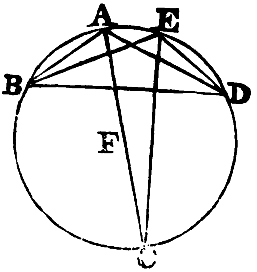
PROP. XIII. THEOR.

The angles in the same segment of a circle are equal to one another.



Let ABCD be a circle, and BAD, BED angles in the same segment BAED; the angles BAD, BED are equal to one another.

Take F the centre of the circle ABCD; and first, let the segment BAED be greater than a semicircle, and join BF, FD; and because the angle BFD is at the centre, and the angle BAD at the circumference, and that they have the same part of the circumference, viz. BCD, for their base; therefore the angle BFD is double (12. 3.) of the angle BAD: for the same reason, the angle BFD is double of the angle BED: therefore the angle BAD is equal to the angle BED.



But, if the segment BAD be not greater than a semicircle, let BAD, BED be angles in it; these also are equal to one another: draw AF to the centre, and produce it to C, and join CE: therefore the segment BADC is greater than a semicircle; and the angles in it, BAC, BEC are equal, by the first case; for the same reason, because CBED is greater than a semicircle, the angles CAD, CED are equal: therefore the whole angle BAD is equal to the whole angle BED. Wherefore the angles in the same segment, &c. Q. E. D.

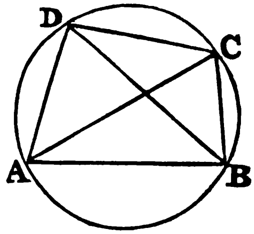
Cor. Segments of circles which contain equal angles are similar.

PROP. XIV. THEOR.

The opposite angles of any quadrilateral figure described in a circle, are together equal to two right angles.

Let ABCD be a quadrilateral figure in the circle ABCD; any two of its opposite angles are together equal to two right angles.

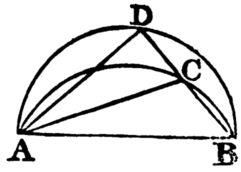
Join AC, BD; and because the three angles of every triangle are equal (5. 2.) to two right angles, the three angles of the triangle CAB, viz. the angles CAB, ABC, BCA are equal to two right angles: but the angle CAB is equal (13. 3.) to the angle CDB, because they are in the same segment BADC, and the angle ACB is equal to the angle ADB, because they are in the same segment ADCB: therefore the whole angle ADC is equal to the angles CAB, ACB: to each of these equals add the angle ABC: therefore the angles ABC, CAB, BCA are equal to the angles ABC, ADC: but ABC, CAB, BCA are equal to two right angles; therefore also the angles ABC, ADC are equal to two right angles; in the same manner, the angles BAD, DCB may be shown to be equal to two right angles. Therefore, the opposite angles, &c. Q. E. D.



PROP. XV. THEOR.

Upon the same straight line, and upon the same side of it, there cannot be two similar segments of circles, not coinciding with another.

If it be possible, let the two similar segments of circles, viz. ACB, ADB be upon the same side of the same straight line AB, not coin-



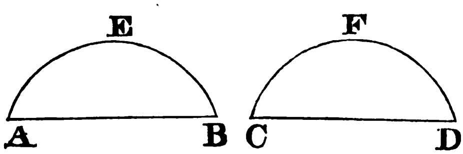
ciding with one another; then, one of the segments must fall within the other: let ACB fall within ADB, and draw the straight line BCD, and join CA, DA: and because the segment ACB is similar to the segment ADB, and that similar segments of circles contain equal angles; the angle ACB is equal to the angle ADB, the exterior to the interior, which is impossible. Therefore, there cannot be two similar segments of a circle upon the same side of the same line, which do not coincide. Q. E. D.

PROP. XVI. THEOR.

Similar segments of circles upon equal straight lines, are equal to one another.

Let AEB, CFD be similar segments of circles upon the equal straight lines AB, CD: the segment AEB is equal to the segment CFD.

For, if the segment AEB be applied to the segment CFD, so as the



point A be on C, and the straight line AB upon CD, the point B shall coincide with the point D, because AB is equal to CD; therefore the straight line AB coinciding with CD, the segment AEB must (15. 3.) coincide with the segment CFD, and therefore is equal to it. Wherefore, similar segments, &c. Q. E. D.

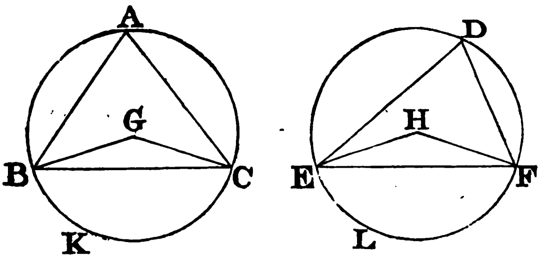
PROP. XVII. THEOR.

In equal circles, equal angles stand upon equal circumferences, whether they be at the centres or circumferences.

Let ABC, DEF be equal circles, and the equal angles BGC, EHF at their centres, and BAC, EDF at their circumferences; the circumference BKC is equal to the circumference ELF.

Join BC, EF; and because the circles ABC, DEF are equal, the straight lines drawn from their centres are equal; therefore the two sides BG, GC, are equal to the two EH, HF; and the angle at G is equal to the angle at H; therefore the base BC is equal (4. 1.) to the base EF; and because the angle at A is equal to the angle at D, the segment BAC is similar (Cor. 13. 3.) to the segment EDF; and they

are upon equal straight lines BC, EF; but similar segments of circles upon equal straight lines are equal (16. 3.) to one another; therefore the segment BAC is equal to the segment EDF; but the whole



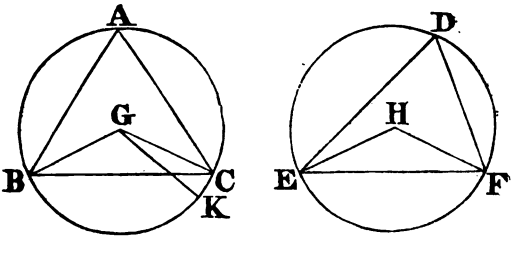
circle ABC is equal to the whole DEF; therefore the remaining segment BKC is equal to the remaining segment ELF; and the circumference BKC to the circumference ELF. Wherefore, in equal circles, &c. Q. E. D.

PROP. XVIII. THEOR.

In equal circles, the angles which stand upon equal circumferences are equal to one another, whether they be at the centres or circumferences.

Let the angles BGC, EHF at the centres, and BAC, EDF at the circumference of the equal circles ABC, DEF stand upon the equal circumferences BC, EF: the angle BGC is equal to the angle EHF, and the angle BAC to the angle EDF.

If the angle BGC be equal to the angle EHF, it is manifest (12. 3.) that the angle BAC is also equal to EDF: but, if not, one of them is the greater, let BGC be the greater: and at the point G, in the straight line BG, make (14. 1.) the angle BGK equal to the angle EHF; but



equal angles stand upon equal circumferences (17. 3.) when they are at the centre; therefore the circumference BK is equal to the circumference EF: but EF is equal to BC; therefore also BK is equal to BC, the less to the greater, which is impossible: therefore the angle BGC is not unequal to the angle EHF; that is, it is equal to it: and

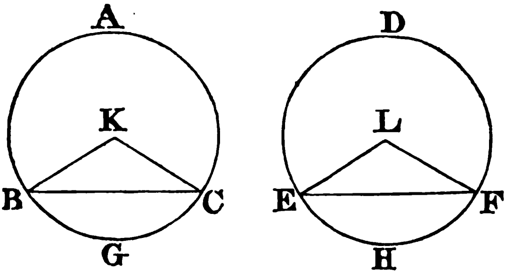
the angle at A is half of the angle BGC, and the angle at D half of the angle EHF: therefore the angle at A is equal to the angle at D. Wherefore, in equal circles, &c. Q. E. D.

PROP. XIX. THEOR.

In equal circles, equal straight lines cut off equal circumferences, the greater equal to the greater, and the less to the less.

Let ABC, DEF be equal circles, and BC, EF equal straight lines in them, which cut off the two greater circumferences BAC, EDF, and the two less BGC, EHF; the greater BAC is equal to the greater EDF, and the less BGC to the less EHF.

Take (3. 3.) K, L the centres of the circles, and join BK, KC, EL, LF; and because the circles are equal, the straight lines from their centres are equal, therefore BK, KC, are equal to EL, LF; and the base BC is equal to the base EF; therefore the angle BKC is equal



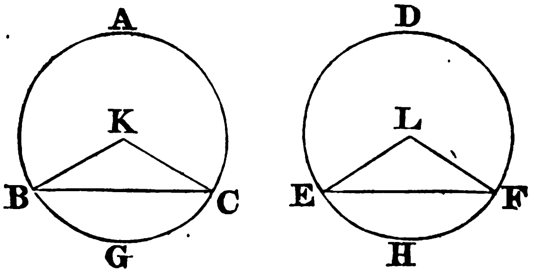
(7. 1.) to the angle ELF; but equal angles stand upon equal (17. 3.) circumferences, when they are at the centres: therefore the circumference BGC is equal to the circumference EHF; but the whole circle ABC is equal to the whole EDF; the remaining part therefore of the circumference, viz. BAC, is equal to the remaining part EDF. Therefore, in equal circles, &c. Q. E. D.

PROP. XX. THEOR.

In equal circles, equal circumferences are subtended by equal straight lines.

Let ABC, DEF be equal circles, and let the circumference BGC, EHF also be equal; and join BC, EF: the straight line BC is equal to the straight line EF.

Take (3. 3.) K, L, the centres of the circles, and join BK, KC, EL, LF; and because the circumference BGC is equal to the circumference EHF, the angle BKC is equal (18. 3.) to the angle ELF: and



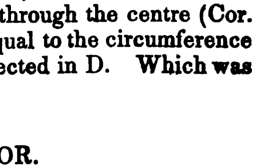
because the circles ABC, DEF are equal, the straight lines from their centres are equal: therefore BK, KC are equal to EL, LF, and they contain equal angles: therefore the base BC is equal (4. 1.) to the base EF. Therefore, in equal circles, &c. Q. E. D.

PROP. XXI. PROB.

To bisect a given circumference, that is, to divide it into two equal parts.

Let ADB be the given circumference; it is required to bisect it.

Join AB, and bisect (8. 1.) it in C; from the point C draw CD at right angles to AB, and join AD, DB: the circumference ADB is bisected in the point D.



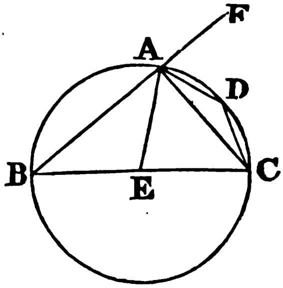
Because AC is equal to CB, and CD common to the triangles ACD, BCD, the two sides AC, CD are equal to the two BC, CD; and the angle ACD is equal to the angle BCD, because each of them is a right angle; therefore the base AD is equal (4. 1.) to the base BD: but equal straight lines cut off equal (19. 3.) circumferences, the greater equal to the greater, and the less to the less, and AD, DB are each of them less than a semicircle; because DC passes through the centre (Cor. 3. 3.): wherefore the circumference AD is equal to the circumference DB: therefore the given circumference is bisected in D. Which was to be done.

PROP. XXII. THEOR.

In a circle, the angles in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Let ABCD be a circle, of which the diameter is BC, and centre E; and draw CA, dividing the circle into the segments ABC, ADC, and join BA, AD, DC; the angle in the semicircle BAC is a right

angle; and the angle in the segment ABC, which is greater than a semicircle, is less than a right angle; and the angle in the segment ADC, which is less than a semicircle, is greater than a right angle.



Join AE, and produce BA to F; and because BE is equal to EA, the angle EAB is equal (5. 1.) to EBA; also, because AE is equal to EC, the angle EAC is equal to ECA; wherefore the whole angle BAC is equal to the two angles ABC, ACB; but FAC, the exterior angle of the triangle ABC, is equal (5. 2.) to the two angles ABC, ACB; therefore the angle BAC is equal to the angle FAC, and each of them is therefore a right (Def. 27. 1.) angle: wherefore the angle BAC in a semicircle is a right angle.

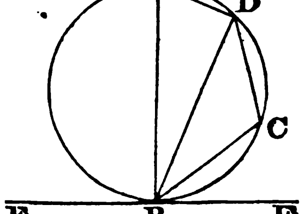
And because the two angles ABC, BAC, of the triangle ABC are together less (5. 2.) than two right angles, and that BAC is a right angle, ABC must be less than a right angle: and therefore the angle in a segment ABC greater than a semicircle, is less than a right angle.

And because ABCD is a quadrilateral figure in a circle, any two of its opposite angles are equal (14. 3.) to two right angles; therefore the angles ABC, ADC are equal to two right angles; and ABC is less than a right angle; wherefore the other ADC is greater than a right angle.

PROP. XXIII. THEOR.

If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angles made by this line with the line touching the circle, shall be equal to the angles which are in the alternate segments of the circle.

Let the straight line EF touch the circle ABCD in B, and from the point B let the straight line BD be drawn cutting the circle; the angles which BD makes with the touching line EF shall be equal to the angles in the alternate segments of the circle; that is, the angle FBD is equal to the angle which is in the segment DAB, and the angle DBE to the angle in the segment BCD.



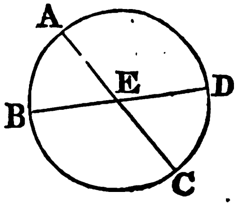
From the point B draw (9. 1.) BA at right angles to EF, and take any point C in the circumference BD, and join AD, DC, CB; and because the straight line EF touches the circle ABCD in the point B, and BA is drawn at right angles to the touching line from the point of contact B, the centre of the circle is (11. 3.) in BA; therefore the angle ADB in a semicircle is a right angle, and con-

sequently the other two angles BAD, ABD are equal (5. 2.) to a right angle; but ABF is likewise a right angle; therefore the angle ABF is equal to the angles BAD, ABD; take from these equals the common angle ABD; therefore the remaining angle DBF is equal to the angle BAD, which is in the alternate segment of the circle; and because ABCD is a quadrilateral figure in a circle, the opposite angles BAD, BCD are equal (14. 3.) to two right angles; therefore the angles DBF, DBE being likewise equal to two right angles, are equal to the angles BAD, BCD; and DBF has been proved equal to BAD; therefore the remaining angle DBE is equal to the angle BCD in the alternate segment of the circle. Wherefore, if a straight line, &c. Q. E. D.

PROP. XXIV. THEOR.

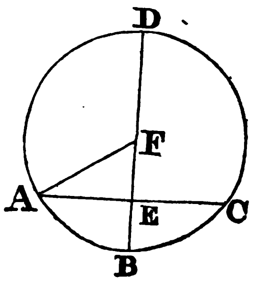
If two straight lines within a circle cut one another, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.

Let the two straight lines AC, BD, within the circle ABCD, cut one another in the point E: the rectangle contained by AE, EC is equal to the rectangle contained by BE, ED.



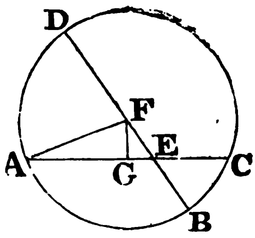
If AC, BD pass each of them through the centre, so that E is the centre; it is evident, that AE, EC, BE, ED, being all equal, the rectangle AE, EC is likewise equal to the rectangle BE, ED.

But let one of them BD pass through the centre, and cut the other AC, which does not pass through the centre, at right angles, in the point E: then, if BD be bisected in F, F is the centre of the circle ABCD; join AF: and because BD, which passes through the centre, cuts the straight line AC, which does not pass through the centre, at right angles in E, AE, EC are equal (5. 3.) to one another: and because the straight line BD is cut into two equal parts in the point F, and into two unequal in the point E, the rectangle BE, ED together with the square of EF, is equal (5. 2.) to the square of FB; that is, to the square of FA; but the squares of AE, EF are equal (17. 2.) to the square of FA; therefore the rectangle BE, ED, together with the square of EF, is equal to the squares of AE, EF: take away the common square of EF, and the remaining rectangle BE, ED is equal to the remaining square of AE; that is, to the rectangle AE, EC.

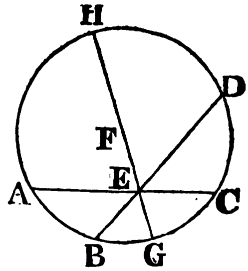


Next, Let BD, which passes through the centre, cut the other AC, which does not pass through the centre, in E, but not at right angles: then, as before, if BD be bisected in F, F is the centre of the circle.

Join AF, and from F draw (10. 1.) FG perpendicular to AC; therefore AG is equal (5. 3.) to GC; wherefore the rectangle AE, EC, together with the square of EG, is equal (21. 2.) to the square of AG: to each of these equals add the square of GF; therefore the rectangle AE, EC, together with the squares of EG, GF, is equal to the squares of AG, GF: but the squares of EG, GF are equal (17. 1.) to the square of EF; and the squares of AG, GF are equal to the square of AF: therefore the rectangle AE, EC, together with the square of EF, is equal to the square of AF; that is, to the square of FB: but the square of FB is equal (21. 2.) to the rectangle BE, ED, together with the square of EF: therefore the rectangle AE, EC, together with the square of EF, is equal to the rectangle BE, ED, together with the square of EF: take away the common square of EF, and the remaining rectangle AE, EC is therefore equal to the remaining rectangle BE, ED.



Lastly, Let neither of the straight lines AC, BD pass through the centre: take the centre F, and through E, the intersection of the straight lines AC, DB, draw the diameter GEFH: and because the rectangle AE, EC is equal, as has been shown, to the rectangle GE, EH; and, for the same reason, the rectangle BE, ED is equal to the same rectangle GE, EH; therefore the rectangle AE, EC is equal to the rectangle BE, ED. Wherefore, if two straight lines, &c. Q. E. D.

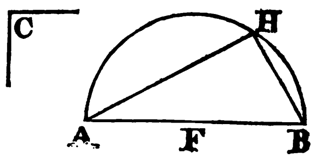


PROP. XXV. PROB.

Upon a given straight line to describe a segment of a circle, containing an angle equal to a given angle.

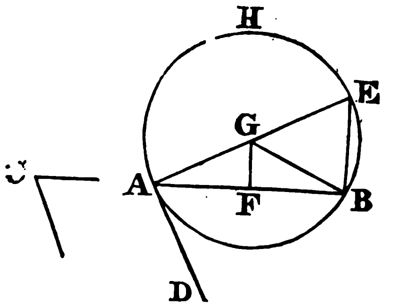
Let AB be the given straight line, and the angle at C the given rectilineal angle; it is required to describe upon the given straight line AB a segment of a circle, containing an angle equal to the angle C.

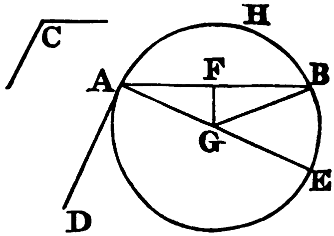
First, Let the angle at C be a right angle, and bisect (8. 1.) AB in F, and from the centre F, at the distance FB, describe the semicircle AHB; therefore the angle AHB in a semicircle is (22. 3.) equal to the right angle at C.



But if the angle C be not a right angle, at the point A in the straight line AB, make (14. 1.) the angle BAD equal to the angle C, and from the point A draw (10. 1.) AE

at right angles to AD; bisect (9. 1.) AB in F, and from F draw (9. 1.) FG at right angles to AB, and join GB: and because AF is equal to FB, and FG common to the triangles AFG, BFG, the two sides AF, FG are equal to the two BF, FG; and the angle AFG is equal to the angle BFG; therefore the base AG is equal (4. 1.) to the base GB; and the circle described from the centre G, with the radius GA, shall pass through the point B: Let this be the circle AHB, and because from the point A the extremity of the diameter AE, AD is drawn at right angles to AE, therefore AD (8. 3.) touches the circle; and because AB drawn from the point of contact A cuts the circle, the angle DAB is equal to the angle in the alternate segment AHB (23. 3.); but the angle DAB is equal to the angle C, therefore also the angle C is equal to the angle in the segment AHB: wherefore upon the given straight line AB the segment AHB of a circle is described which contains an angle equal to the given angle at C. Which was to be done.



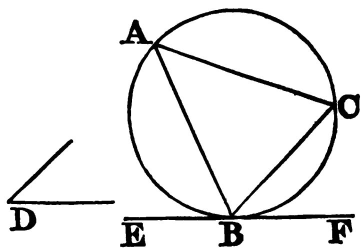


PROP. XXVI. PROB.

To cut off a segment from a given circle which shall contain an angle equal to a given rectilineal angle.

Let ABC be the given circle, and D the given rectilineal angle; it is required to cut off a segment from the circle ABC that shall contain an angle equal to the given angle D.

Draw (9. 3.) the straight line EF touching the circle ABC in the point B, and at the point B, in the straight line BF, make (14. 1.) the angle FBC equal to the angle D; therefore, because the straight line EF touches the circle ABC, and BC is drawn from the point of contact B, the angle FBC is equal (23. 3.) to the angle in the alternate segment BAC of the circle: but the angle FBC is



equal to the angle D; therefore the angle in the segment BAC is equal to the angle D: wherefore the segment BAC is cut off from the given circle ABC, containing an angle equal to the given angle D. Which was to be done.

THE

ELEMENT

OF

GEOMETRY.

BOOK IV.

DEFINITIONS.

I.

The relation of one magnitude to another of the same kind in respect of quantity, may be called a ratio.

II.

Each of the quantities in a ratio may be called a term.

III.

If the ratio of the first to the second is the same with that of the third to the fourth, the quantities may be said to be in geometrical proportion.

IV.

In treating of proportion it is usual to employ a number, line, or letter, to express the quantity, and to consider the quantity without reference to kind.

V.

Proportion consists of two ratios, and each ratio of two terms; there may be, however, only three quantities, the second being repeated.

VI.

And the second term may then be called a mean proportional.

VII.

In proportion the first and third terms may be called antecedents.

VIII.

And the second and fourth consequents.

IX.

The antecedents may be said to be homologous to each other.

X.

Likewise the consequents.

XI.

The first and fourth terms may be called extremes.

XII.

And the second and third, means.

XIII.

If the ratio of every two adjacent terms is the same, the proportion may be said to be continued.

XIV.

Continued proportion may be called progression.

XV.

Whatever is the ratio of the first to the second, and of the second to the third, the ratio of the first to the third is compounded of the ratio of the first to the second, and of the second to the third. And when the ratio of the second to the third, is the same with that of the first to the second, the ratio of the first to the second may be considered as repeated, and therefore if three quantities are in continued proportion, the first may be said to have to the third, the duplicate ratio of that which it has to the second.

XVI.

If four, the triplicate ratio, and so on, increasing the denomination still by unity in any number of proportionals.

XVII.

A ratio may be thus expressed, A : B ; as A is to B.

XVIII.

A ratio may also be expressed by the part which one quantity is of another; as, $\frac{A}{B}$; A divided by B.

XIX.

Proportion may be thus expressed ; A : B : : C : D. As A is to B, so is C to D.

XX.

Or the two equal ratios which make up the proportion may be thus expressed, $\frac{A}{B} = \frac{C}{D}$; A divided by B is equal to C divided by D.

PROPOSITION I. THEOREM.

If four quantities are proportional, the product of the two extremes is equal to the product of the two means.

Let A : B : : C : D; then AD=BC.

Because A : B : : C : D, $\frac{A}{B}=\frac{C}{D}$ (Def. 20. 4.); multiply both by BD, and $\frac{ABD}{B}=\frac{CBD}{D}$, or AD=BC.

PROP II. THEOR.

If four quantities are such that the product of two is equal to the product of the other two, these quantities are proportional.

Let AD=BC; then A : B : : C : D.

Because AD=BC, divide both by BD and $\frac{AD}{BD}=\frac{BC}{BD}$, or $\frac{A}{B}=\frac{C}{D}$; and therefore A : B : : C : D.

PROP. III. THEOR.

If four quantities are proportional, the product of the means divided by either extreme, will give the other extreme.

Let A : B : : C : D, $\frac{BC}{A}=D$.

Because A : B : : C : D, AD=BC; divide by A, and $D=\frac{BC}{A}$.

PROP. IV. THEOR.

The products of the corresponding terms of two proportions are also proportional.

Let A : B : : C : D, and E : F : : G : H; then AE : BF : : CG : DH.

Because A : B : : C : D, and E : F : : G : H, $\frac{A}{B}=\frac{C}{D}$, and $\frac{E}{F}=\frac{G}{H}$; multiply $\frac{A}{B}$ by $\frac{E}{F}$, and $\frac{C}{D}$ by $\frac{G}{H}$ equal to $\frac{E}{F}$, and $\frac{A}{B}\times\frac{E}{F}=\frac{C}{D}\times\frac{G}{H}$; hence $\frac{AE}{BF}=\frac{CG}{DH}$, and therefore AE : BF : : CG : DH.

Cor. If four quantities are proportional, their squares and cubes will likewise be proportional.

PROP. V. THEOR.

If four quantities, A, B, C, D *are proportional; then*,

Inversely, B : A : : D : C.

VI. Alternately, A : C : : B : D.

VII. Compoundedly, A : A+B : : C : C+D.

VIII. Dividedly, A : A—B : : C : C—D.

IX. Mixtly, B+A : B—A : : D+C : D—C.

X. By Multiplication, RA : RB : : C : D.

XI. By Division, $\frac{A}{R} : \frac{B}{R} :: C : D$.

Because in each case the product of the means is equal to that of the extremes, and therefore the quantities are proportional.

PROP. XII. THEOR.

In geometrical progression, the product of the extremes is equal to the product of any two terms equidistant from them.

Let A, B, C, D, E, F be in progression ; then AF=BE or CD.

Because the ratio of every two adjacent terms in progression is the same (Def. 13. 4.), A : B : : E : F, and therefore AF=BE ; and it may be shewn in the same way, that AF=CD ; and therefore AF=BE or CD.

PROP. XIII. THEOR.

Equal quantities have the same ratio to the same; and the same has the same ratio to equal quantities.

Let A and B be equal quantities, and C any other quantity; as A : C : : B : C.

Because A is equal to B, C is the same part of A that it is of B, and therefore $\frac{A}{C}=\frac{B}{C}$; or A : C : : B : C.

Likewise C : A : : C : B.

Because A is equal to B, A is the same part of C that B is of C, and therefore $\frac{C}{A}=\frac{C}{B}$; or C : A : : C : B.

PROP. XIV. THEOR.

Quantities which have the same ratio to the same are equal; and those to which the same quantity has the same ratio are equal.

Let A : C : : B : C; A=B.

Because A : C : : B : C, $\frac{A}{C}=\frac{B}{C}$ (Def. 20. 4.). Multiply by C, and A=B.

Likewise, Let C : A : : C : B; A=B.

Because C : A : : C : B, $\frac{C}{A}=\frac{C}{B}$ (Def.20.4.). Divide by C, and A=B.

PROP. XV. THEOR.

Ratios that are the same to the same ratio, are the same to one another.

Let A : B : : C : D, and C : D : : E : F; A : B : : E : F.

Because A : B : : C : D, $\frac{A}{B}=\frac{C}{D}$ (Def. 20. 4.); and because C : D : : E : F, $\frac{C}{D}=\frac{E}{F}$; and therefore $\frac{A}{B}=\frac{E}{F}$, or A : B : : E : F.

PROP. XVI. THEOR.

If there be any number of quantities, and as many others, which taken two and two in order have the same ratio, the first shall have to the last of the first quantities the same ratio which the first of the others has to the last. This is usually cited by the words "ex æquali," or "ex æquo."

First, Let there be three quantities A, B, C, and three others D, E, F; and let A : B : : D : E, and B : C : : E : F; then A : C : : D : F.

Because A : B : : D : E, $\frac{A}{B}=\frac{D}{E}$; and because B : C : : E : F, $\frac{B}{C}=\frac{E}{F}$; multiply $\frac{A}{B}$ by $\frac{B}{C}$, also $\frac{D}{E}$ by $\frac{E}{F}$ equal to $\frac{B}{C}$, and $\frac{AB}{BC}=\frac{DE}{EF}$; or $\frac{A}{C}=\frac{D}{F}$; or A : C : : D : F.

Next, Let there be four quantities A, B, C, D, and four others E, F, G, H; and let A : B : : E : F, and B : C : : F : G, and C : D : : G : H; A : D : : E : H.

Because A, B, C are three quantities, and E, F, G three others,

which taken two and two have the same ratio, by the above case, A : C :: E : G; but C : D :: G : H, therefore again by the above case, A : D :: E : H; and so on whatever be the number of quantities.

PROP. XVII. THEOR.

If there be any number of quantities, which taken two and two in a cross order have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last. This is usually cited by the words "ex æquali in proportione perturbata," or "ex æquo perturbato."

First, Let there be three quantities A, B, C, and D, E, F, and let A : B :: E : F, and B : C :: D : E; then A : C :: D : F.

Because A : B :: E : F, $\frac{A}{B}=\frac{E}{F}$ (Def. 20. 4.); and because B : C :: D : E, $\frac{B}{C}=\frac{D}{E}$ (Def. 20. 4.); multiply $\frac{A}{B}$ by $\frac{B}{C}$ and $\frac{E}{F}$ by $\frac{D}{E}$ equal to BC, and $\frac{AB}{BC}=\frac{ED}{FE}$, or $\frac{A}{C}=\frac{D}{F}$, or A : C :: D : F (Def. 19. 4.).

Next, Let there be four quantities A, B, C, D, and four others E, F, G, H; and let A : B :: G : H, B : C :: F : G, and C : D :: E : F; then A : D :: E : H.

Because A, B, C, are three quantities, and F, G, H, three others, which taken two and two in a cross order have the same ratio, by the first case, A : C :: F : H, but C : D :: E : F, wherefore again by the first case, A : D :: E : H; and so on whatever be the number of magnitudes. Therefore, If, &c. Q. E. D.

PROP. XVIII. THEOR.

If the first has to the second the same ratio which the third has to the fourth; and the fifth to the second, the same ratio which the sixth has to the fourth; the first and fifth together shall have to the second, the same ratio which the third and sixth together have to the fourth.

Let A : B :: C : D, and E : B :: F : D; then A+E : B :: C+F : D.

Because A : B :: C : D, $\frac{A}{B}=\frac{C}{D}$ (Def. 20. 4.); and because E : B :: F : D, $\frac{E}{B}=\frac{F}{D}$ (Def. 20. 4.); to $\frac{A}{B}$ add $\frac{E}{B}$, and to $\frac{C}{D}$ equal to $\frac{A}{B}$ add $\frac{F}{D}$ equal to $\frac{E}{B}$, and $\frac{A}{B}+\frac{E}{B}=\frac{C}{D}+\frac{F}{D}$, or $\frac{A+E}{B}=\frac{C+F}{D}$, and therefore A+E : B :: C+F : D. Therefore, If, &c. Q. E. D.

Cor. I. If the same hypothesis be made as in the proposition, the excess of the first and fifth shall be to the second, as the excess of the

third and sixth to the fourth. The demonstration of this is the same with that of the proposition, if division be used instead of composition.

Cor. II. The proposition holds true of two ranks of magnitudes, whatever be their number, of which each of the first rank has to the second magnitude the same ratio that the corresponding one of the second rank has to a fourth magnitude; as is manifest.

THE ELEMENT OF GEOMETRY.

BOOK V.

DEFINITIONS.

I.

Two magnitudes may be said to be reciprocally proportional to two others, when one of the first is to one of the second, as the remaining one of the second is to the remaining one of the first.

II.

A straight line may be said to be cut in extreme and mean ratio, when the whole is to the greater segment, as the greater segment is to the less.

III.

When two magnitudes can be divided into any number of equal parts, so that a part of one is equal to a part of the other, the magnitudes may be said to be commeasurable.

IV.

When they cannot be so divided, they may be said to be incommeasurable.

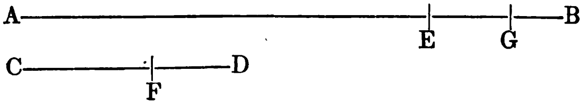
V.

Rectilineal figures which have their several angles equal, each to each, and the sides about the equal angles proportional, differ only in size, and therefore are the same in shape or similar.

PROP. I. PROBLEM.

To find the greatest common measure of two given straight lines.

Let AB and CD be the two given straight lines, it is required to find their greatest common measure. Apply the less CD to the



greater, as many times as it can be contained therein, for instance, twice, with the remainder BE ; apply the remainder BE in the same way to CD, for instance, once, with the second remainder DF; apply DF in the same way to BE, for instance, once, with the third remainder BG; apply the third remainder BG to the second remainder DF; and continue thus until a remainder is found which is contained in the preceding without leaving a remainder.

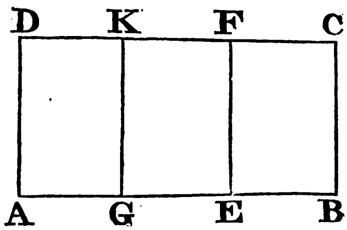
Because the less line CD is to be measured, the common measure cannot be greater than CD, and if CD measures AB without a remainder, it is the greatest common measure. But if a remainder is left as BE, then the greatest common measure is less than CD, and not greater than BE ; and if BE measures CD without a remainder, BE is the greatest common measure ; and it may be shewn in the same way, that the first remainder that measures the preceding without a remainder, is the greatest common measure.

PROP. II. THEOR.

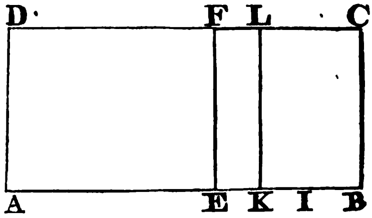
Parallelograms of the same altitude are to one another as their bases.

Let ABCD and AEFD, be two parallelograms which have the same altitude, they are to each other as their bases AB, AE.

If AB and AE are commeasurable, let AG be the common measure ; divide AB and AE into the equal parts AG, GE, and EB, and draw GK, EF, parallel to AD or BC (4. 2.) ; then the parallelograms thus made will be equal, and the parallelograms ABCD and AEFD will be to each other as the respective numbers of the equal parallelograms of which they are composed ; and because the bases have been divided into equal parts, these numbers will be to each other as the bases ; and therefore the parallelograms ABCD, AEFD, will be to each other as the bases.



But if the bases AB and AE are not commeasurable, still ABCD : AEFD : : AB : AE.



For if not, let ABCD : AEFD : : AB : AI, longer than AE; divide AB into equal parts less than EI, and let one of the divisions fall between E and I at K, and draw KL parallel to BC: then AB and AK being commeasurable, ABCD : AKLD : : AB : AK; but by the hypothesis, ABCD : AEFD : : AB : AI; and in these two proportions the antecedents being equal, the consequents are proportional (13. 4.); and therefore, AKLD : AEFD : : AK : AI, but AI is longer than AK, and therefore AEFD must be larger than AKLD, whereas it is less; and therefore ABCD is not to AEFD as AB to AI, any line longer than AE. It may be shewn in the same way, that ABCD is not to AEFD as AB to any line shorter than AE; and therefore, ABCD : AEFD : : AB : AE. Therefore, Parallelograms, &c. Q. E. D.

Cor. I. Because parallelograms of equal base and altitude are equal, therefore parallelograms of equal altitude are to one another as their bases.

Cor. II. Because a triangle is equal to half a parallelogram of equal base and altitude, therefore triangles of equal altitude are to one another as their bases.

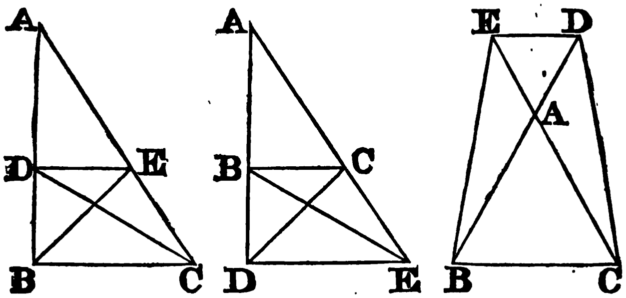
PROP. III. THEOR.

If a straight line be drawn parallel to one of the sides of a triangle, it shall cut the other sides, or these produced, proportionally; and if the sides, or the sides produced be cut proportionally, the straight line which joins the points of section shall be parallel to the remaining side of the triangle.

Let DE be drawn parallel to BC, one of the sides of the triangle ABC; BD is to DA, as CE to EA.

Join BE, CD; then the triangle BDE is equal to the triangle CDE (Cor. 1. 15. 2.), because they are on the same base DE, and between the same parallels DE, BC. ADE is another triangle, and equal magnitudes have to the same, the same ratio (13. 4.); therefore as the triangle BDE to the triangle ADE, so is the triangle CDE to the triangle ADE; but as the triangle BDE to the triangle ADE, so is (Cor. 2. 2. 5.) BD to DA, because having the same altitude, viz. the perpendicular drawn from the point E to AB, they are to one another as their bases; and for the same reason, as the triangle CDE to the triangle ADE, so is CE to EA: therefore as BD to DA, so is CE to EA.

Next, let the sides AB, AC of the triangle ABC, or these produced,



be cut proportionally in the points D, E, that is, so that BD be to DA, as CE to EA, and join DE; DE is parallel to BC.

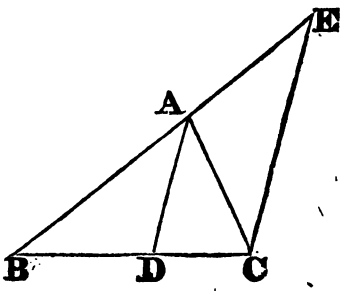
The same construction being made; because as BD to DA, so is CE to EA; and as BD to DA, so is the triangle BDE to the triangle ADE (2. Cor. 2. 5.); and as CE to EA, so is the triangle CDE to the triangle ADE; therefore the triangle BDE is to the triangle ADE, as the triangle CDE to the triangle ADE; that is, the triangles BDE, CDE have the same ratio to the triangle ADE; and therefore (14. 4.) the triangle BDE is equal to the triangle CDE; and they are on the same base DE; but equal triangles on the same base are between the same parallels (2. Cor. 15. 2.), therefore DE is parallel to BC. Wherefore, if a straight line, &c. Q. E. D.

PROP. IV. THEOR.

If the angle of a triangle be divided into two equal angles, by a straight line which also cuts the base; the segments of the base shall have the same ratio which the other sides of the triangle have to one another: and if the segments of the base have the same ratio which the other sides of the triangle have to one another, the straight line drawn from the vertex to the point of section, divides the vertical angle into two equal angles.

Let the angle BAC of any triangle ABC be divided into two equal angles by the straight line AD: BD is to DC, as BA to AC.

Through the point C draw CE parallel (4. 2.) to DA, and let BA produced meet CE in E. Because the straight line AC meets the parallels AD, EC, the angle ACE is equal to the corresponding angle CAD (3. 2.); but CAD by the hypothesis is equal to the angle BAD; wherefore BAD is equal to the angle ACE. Again, because the straight line BAE meets the parallels AD, EC, the outward angle BAD is



equal to the inward and opposite angle AEC: but the angle ACE has been proved equal to the angle BAD; therefore also ACE is equal to the angle AEC, and consequently the side AE is equal to the side (6. 1.) AC; and because AD is drawn parallel to one of the sides of the triangle BCE, viz. to EC, BD is to DC, as BA to AE (3. 5.): but AE is equal to AC; therefore, as BD to DC, so is BA to AC (13. 4.).

Let now BD be to DC, as BA to AC, and join AD; the angle BAC is divided into two equal angles by the straight line AD.

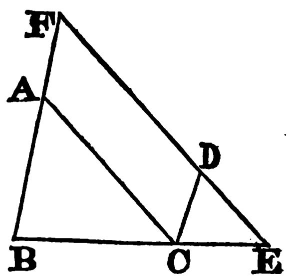
The same construction being made; because, as BD to DC, so is BA to AC: and as BD to DC, so is BA to AE (3. 5.), because AD is parallel to EC; therefore BA is to AC, as BA to AE (13. 4.): consequently AC is equal to AE (14. 4.), and the angle AEC is therefore equal to the angle ACE (5. 1.): but the angle AEC is equal to the corresponding angle BAD: and the angle ACE is equal to the corresponding angle CAD (3. 2.): wherefore also the angle BAD is equal to the angle CAD: therefore the angle BAC is cut into two equal angles by the straight line AD. Therefore, if the angle, &c. Q. E. D.

PROP. V. THEOR.

The sides about the equal angles of equiangular triangles are proportionals; and those which are opposite to the equal angles are homologous sides, that is, are the antecedents or consequents of the ratios.

Let ABC, DCE be equiangular triangles, having the angle ABC equal to the angle DCE, and the angle ACB to the angle DEC, and consequently (5. 2.) the angle BAC equal to the angle CDE. The sides about the equal angles of the triangles ABC, DCE are proportionals; and those are the homologous sides which are opposite to the equal angles.

Let the triangle DCE be placed so that its side CE may be contiguous to BC, and in the same straight line with it; and because the angles ABC, ACB are together less than two right angles (5. 2.); ABC and DEC, which is equal to ACB, are also less than two right angles: wherefore BA, ED produced shall meet (13. 2.); let them be produced and meet in the point F, and because the angle ABC is equal to the angle DCE, BF is parallel (1. Cor. 2. 2.) to CD. Again, because the angle ACB is equal to the angle DEC, AC is parallel to FE (1. Cor. 2. 2.); therefore FACD is a parallelogram; and consequently AF is equal to CD, and AC to FD (9. 2.): and because AC is parallel to FE, one of the sides of the triangle FBE, BA is to AF, as BC to CE (3. 5.): but AF is equal to CD, therefore (13. 4.) as BA to CD, so is BC to CE; and alternately, as



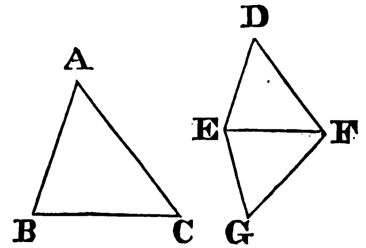
AB to BC, so DC to CE. Again, because CD is parallel to BF, as BC to CE, so is FD to DE (3. 5.); but FD is equal to AC; therefore as BC to CE, so is AC to DE; and alternately, as BC to CA, so CE to ED: therefore because it has been proved that AB is to BC, as DC to CE; and as BC to CA, so CE to ED, ex æquali (16. 4.), BA is to AC, as CD to DE. Therefore the sides, &c. Q. E. D.

PROP. VI. THEOR.

If the sides of two triangles, about each of their angles, be proportionals, the triangles shall be equiangular, and have their equal angles opposite to the homologous sides.

Let the triangles ABC, DEF have their sides proportionals, so that AB is to BC, as DE to EF; and BC to CA, as EF to FD; and consequently, ex æquali, BA to AC, as ED to DF; the triangle ABC is equiangular to the triangle DEF, and their equal angles are opposite to the homologous sides, viz. the angle ABC equal to the angle DEF, and BCA to EFD, and also BAC to EDF.

At the points E, F, in the straight line EF, make (14. 1.) the angle FEG equal to the angle ABC, and the angle EFG equal to BCA; wherefore the remaining angle BAC is equal to the remaining angle EGF (5. 2.), and the triangle ABC is therefore equiangular to the triangle GEF; and consequently they have their sides opposite to the equal angles proportionals (5. 5.). Wherefore, as AB to BC, so is GE to EF; but as AB to BC, so is DE to EF; therefore as DE to EF, so (15. 4.) GE to EF: therefore DE and GE have the same ratio to EF, and consequently are equal (14. 4.): for the same reason, DF is equal to FG: and because in the triangles DEF, GEF, DE is equal to EG, and EF common, the two sides DE, EF are equal to the two GE, EF, and the base DF is equal to the base GF: therefore the angle DEF is equal (7. 1.) to the angle GEF, and the other angles to the other angles which are subtended by the equal sides (4. 1.): wherefore the angle DFE is equal to the angle GFE, and EDF to EGF: and because the angle DEF is equal to the angle GEF, and GEF to the angle ABC; therefore the angle ABC is equal to the angle DEF: for the same reason the angle ACB is equal to the angle DFE, and the angle at A to the angle at D. Therefore the triangle ABC is equiangular to the triangle DEF. Wherefore, if the sides, &c. Q. E. D.

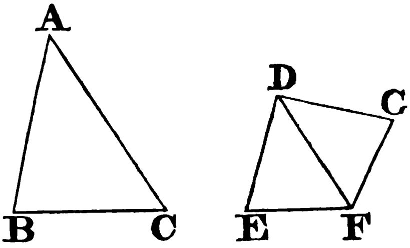


PROP. VII. THEOR.

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles shall be equiangular, and shall have those angles equal which are opposite to the homologous sides.

Let the triangles ABC, DEF have the angle BAC in the one equal to the angle EDF in the other, and the sides about those angles proportionals; that is, BA to AC, as ED to DF; the triangles ABC, DEF are equiangular, and have the angle ABC equal to the angle DEF, and ACB to DFE.

At the points D, F, in the straight line DF, make (14. 1.) the angle FDG equal to either of the angles BAC, EDF; and the angle DFG equal to the angle ACB; wherefore the remaining angle at B is equal to the remaining one at G (5. 2.), and consequently the triangle ABC is equiangular to the triangle DGF; and therefore as BA to AC, so is (5. 5.) GD to DF; but, by the hypothesis, as BA to AC, so is ED to DF; as therefore ED to DF, so is (15. 4.) GD to DF; wherefore ED is equal (14. 4.) to DG; and DF is common to the two triangles EDF, GDF; therefore the two sides ED, DF are equal to the two sides GD, DF: and the angle EDF is equal to the angle GDF; wherefore the base EF is equal to the base FG (4. 1.), and the triangle EDF to the triangle GDF, and the remaining angles to the remaining angles, each to each, which are subtended by the equal sides; therefore the angle DFG is equal to the angle DFE, and the angle at G to the angle at E: but the angle DFG is equal to the angle ACB; therefore the angle ACB is equal to the angle DFE: and the angle BAC is equal to the angle EDF (Hyp.); wherefore also the remaining angle at B is equal to the remaining angle at E. Therefore the triangle ABC is equiangular to the triangle DEF. Wherefore, if two triangles, &c. Q. E. D.



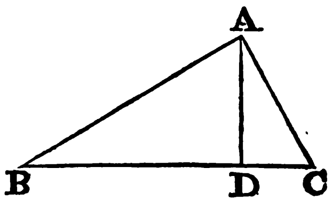
PROP. VIII. THEOR.

In a right angled triangle, if a perpendicular be drawn from the right angle to the base, the triangles on each side of it are similar to the whole triangle, and to one another.

Let ABC be a right angled triangle, having the right angle BAC; and from the point A let AD be drawn perpendicular to the base BC:

the triangles ABD, ADC are similar to the whole triangle ABC, and to one another.

Because the angle BAC is equal to the angle ADB, each of them being a right angle, and that the angle at B is common to the two triangles ABC, ABD; the remaining angle ACB is equal to the remaining angle BAD (5. 2.): therefore the triangle ABC is equiangular to the triangle ABD, and the sides about their equal angles are proportionals (5. 5.); wherefore the triangles are similar (5 Def. 5.): in the like manner it may be demonstrated, that the triangle ADC is equiangular and similar to the triangle ABC: and the triangles ABD, ADC, being both equiangular and similar to ABC, are equiangular and similar to each other. Therefore, in a right angled, &c. Q. E. D.



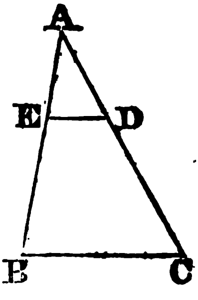
Cor. From this it is manifest, that the perpendicular drawn from the right angle of a right angled triangle to the base, is a mean proportional between the segments of the base: and also that each of the sides is a mean proportional between the base, and its segment adjacent to that side: because in the triangles BDA, ADC, BD is to DA as DA to DC (5. 5.); and in the triangles ABC, DBA, BC is to BA, as BA to BD (5. 5.); and in the triangles ABC, ACD, BC is to CA as CA to CD (5. 5.).

PROP. IX. PROB.

From a given straight line to cut off any part required.

Let AB be the given straight line; it is required to cut off any part from it.

From the point A draw a straight line AC making any angle with AB; and in AC take any point D, and take AC the same multiple of AD, that AB is of the part which is to be cut off from it: join BC, and draw DE parallel to it: then AE is the part required to be cut off.



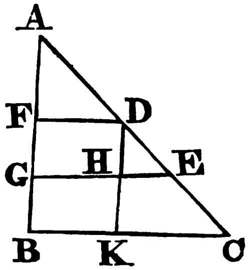
Because ED is parallel to one of the sides of the triangle ABC, viz. to BC, as CD is to DA, so is (3. 5.) BE to EA; and, by composition (7. 4.) CA is to AD as BA to AE: but CA is a multiple of AD; therefore (Def. 18. 4.) BA is the same multiple of AE: whatever part therefore AD is of AC, AE is the same part of AB: wherefore, from the straight line AB the part required is cut off. Which was to be done.

PROP. X. PROB.

To divide a given straight line similarly to a given divided straight line, that is, into parts that shall have the same ratios to one another which the parts of the divided given straight line have.

Let AB be the straight line given to be divided, and AC the divided line; it is required to divide AB similarly to AC.

Let AC be divided in the points D, E; and let AB, AC be placed so as to contain any angle, and join BC, and through the points D, E draw (4. 2.) DF, EG parallels to it; and through D draw DHK parallel to AB: therefore each of the figures FH, HB, is a parallelogram; wherefore DH is equal (9. 2.) to FG, and HK to GB: and because HE is parallel to KC, one of the sides of the triangle DKC, as CE to ED, so is (3. 5.) KH to HD; but KH is equal to BG, and HD to GF; therefore as CE to ED, so is BG to GF: again, because FD is parallel to EG, one of the sides of the triangle AGE, as ED to DA, so is GF to FA; but it has been proved that CE is to ED, as BG to GF; as therefore CE to ED, so is BG to GF; and as ED to DA, so GF to FA: therefore the given straight line AB is divided similarly to AC. Which was to be done.



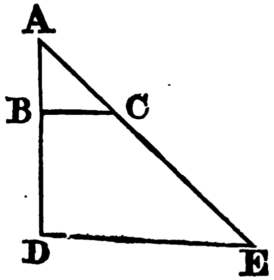
PROP. XI. PROB.

To find a third proportional to two given straight lines.

Let AB, AC be the two given straight lines, and let them be placed so as to contain any angle; it is required to find a third proportional to AB, AC.

Produce AB, AC to the points D, E; and make BD equal to AC, and having joined BC, through D draw DE parallel to it (4. 2.).

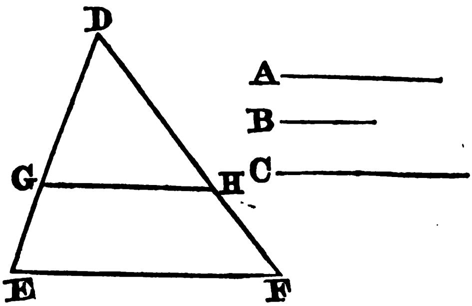
Because BC is parallel to DE, a side of the triangle ADE, AB is (3. 5.) to BD, as AC to CE: but BD is equal to AC; as therefore AB to AC, so is AC to CE. Wherefore to the two given straight lines AB, AC, a third proportional CE is found. Which was to be done.



PROP. XII. PROB.

To find a fourth proportional to three given straight lines.

Let A, B, C be the three given straight lines; it is required to find a fourth proportional to A, B, C.



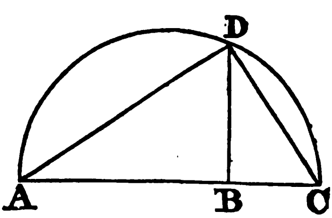
Take two straight lines DE, DF containing any angle EDF; and upon these make DG equal to A, GE equal to B, and DH equal to C; and having joined GH, draw EF parallel (4. 2.) to it through the point E: and because GH is parallel to EF, one of the sides of the triangle DEF, DG is to GE, as DH to HF (3. 5.); but DG is equal to A, GE to B, and DH to C; therefore, as A is to B, so is C to HF: wherefore to the three given straight lines A, B, C, a fourth proportional HF is found. Which was to be done.

PROP. XIII. PROB.

To find a mean proportional between two given straight lines.

Let AB, BC be the two given straight lines; it is required to find a mean proportional between them.

Place AB, BC in a straight line, and upon AC describe the semicircle ADC, and from the point B draw (9. 1.) BD at right angles to AC, and join AD, DC.



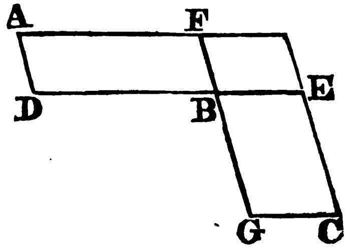
Because the angle ADC in a semicircle is a right angle (22. 3.), and because in the right angled triangle ADC, DB is drawn from the right angle perpendicular to the base, DB is a mean proportional between AB, BC, the segments of the base (Cor. 8. 5.): therefore between the two given straight lines AB, BC a mean proportional DB is found. Which was to be done.

PROP. XIV. THEOR.

Equal parallelograms which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional; and parallelograms that have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Let AB, BC be equal parallelograms which have the angles at B equal, and let the sides DB, BE be placed in the same straight line; wherefore also FB, BG are in one straight line (2. 1.): the sides of the parallelograms AB, BC about the equal angles, are reciprocally proportional; that is, DB is to BE, as GB to BF.

Complete the parallelogram FE; and because the parallelogram AB is equal to BC, and that FE is another parallelogram, AB is to FE, as BC to FE (13. 4.): but as AB to FE, so is the base DB to BE (2. 5.); and as BC to FE, so is the base GB to BF; therefore as DB to BE, so is GB to BF (15. 4.). Wherefore the sides of the parallelograms AB, BC about their equal angles are reciprocally proportional.



But let the sides about the equal angles be reciprocally proportional, viz. as DB to BE, so GB to BF; the parallelogram AB is equal to the parallelogram BC.

Because as DB is to BE, so GB to BF; and as DB to BE, so is the parallelogram AB to the parallelogram FE; and as GB to BF, so is parallelogram BC to parallelogram FE; therefore as AB to FE, so BC to FE (15. 4.): wherefore the parallelogram AB is equal (14. 4.) to the parallelogram BC. Therefore equal parallelograms, &c. Q. E. D.

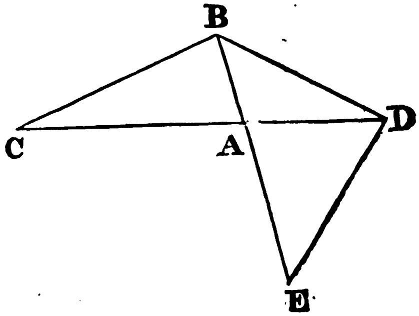
PROP. XV. THEOR.

Equal triangles, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional; and triangles which have one angle in the one equal to one angle in the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Let ABC, ADE be equal triangles, which have the angle BAC equal to the angle DAE; the sides about the equal angles of the triangles are reciprocally proportional; that is, CA is to AD, as EA to AB.

Let the triangles be placed so that their sides CA, AD be in one straight line; wherefore also EA and AB are in one straight line

(2. 1.), and join BD. Because the triangle ABC is equal to the triangle ADE, and that ABD is another triangle, therefore as the triangle CAB is to the triangle BAD, so is the triangle EAD to the triangle DAB (13. 4.): but as the triangle CAB to the triangle BAD, so is the base CA to AD (2. Cor. 2. 5.); and as the triangle EAD to the triangle DAB, so is the base EA to AB (2. Cor. 2. 5.): as therefore CA to AD, so is EA to AB (15. 4.); wherefore the sides of the triangles ABC, ADE about the equal angles are reciprocally proportional.



But let the sides of the triangles ABC, ADE about the equal angles be reciprocally proportional, viz. CA to AD, as EA to AB; the triangle ABC is equal to the triangle ADE.

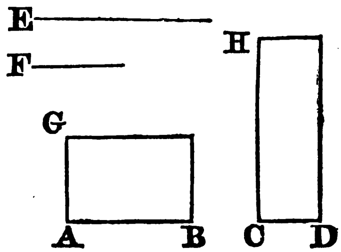
Having joined BD as before; because as CA to AD so is EA to AB; and as CA to AD, so is the triangle BAC to the triangle BAD (2. Cor. 2. 5.); and as EA to AB, so is the triangle EAD to the triangle BAD (2. Cor. 2. 5.); therefore (15. 4.) as the triangle BAC to the triangle BAD, so is the triangle EAD to the triangle BAD; that is, the triangles BAC, EAD have the same ratio to the triangle BAD; wherefore the triangle ABC is equal (14. 4.) to the triangle ADE. Therefore, equal triangles, &c. Q. E. D.

PROP. XVI. THEOR.

If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means; and if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportionals.

Let the four straight lines AB, CD, E, F be proportionals, viz. as AB to CD, so E to F; the rectangle contained by AB, F is equal to the rectangle contained by CD, E.

From the points A, C draw (9. 1.) AG, CH at right angles to AB, CD; and make AG equal to F, and CH equal to E, and complete the parallelograms BG, DH; because as AB to CD, so is E to F; and that E is equal to CH, and F to AG; AB is (13. 4.) to CD, as CH to AG: therefore the sides of the parallelograms BG, DH about the equal angles are reciprocally proportional; but parallelograms which have their



sides about equal angles reciprocally proportional, are equal to one another (14. 5.); therefore the parallelogram BG is equal to the parallelogram DH; and the parallelogram BG is contained by the straight lines AB, F, because AG is equal to F; and the parallelogram DH is contained by CD and E, because CH is equal to E; therefore the rectangle contained by the straight lines AB, F is equal to that which is contained by CD and E.

And if the rectangle contained by the straight lines AB, F be equal to that which is contained by CD, E; these four lines are proportionals, viz. AB is to CD, as E to F.

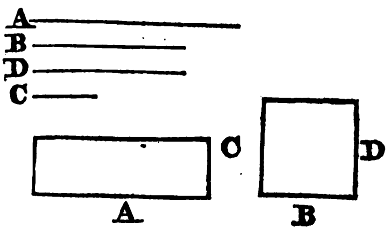
The same construction being made, because the rectangle contained by the straight lines AB, F is equal to that which is contained by CD, E, and that the rectangle BG is contained by AB, F, because AG is equal to F; and the rectangle DH by CD, E, because CH is equal to E; therefore the parallelogram BG is equal to the parallelogram DH; and they are equiangular: but the sides about the equal angles of equal parallelograms are reciprocally proportional (14. 5.); wherefore, as AB to CD, so is CH to AG; and CH is equal to E, and AG to F: as therefore AB is to CD, so E to F. Wherefore, if four, &c. Q. E. D.

PROP. XVII. THEOR.

If three straight lines be proportionals, the rectangle contained by the extremes is equal to the square of the mean: and if the rectangle contained by the extremes be equal to the square of the mean, the three straight lines are proportionals.

Let the three straight lines A, B, C be proportionals, viz. as A to B, so B to C; the rectangle contained by A, C is equal to the square of B.

Take D equal to B; and because as A to B, so B to C, and that B is equal to D; A is (13. 4.) to B, as D to C; but if four straight lines be proportionals, the rectangle contained by the extremes is equal to that which is contained by the means (16. 6.): therefore the rectangle contained by A, C is equal to that contained by B, D. But the rectangle contained by B, D is the square of B; because B is equal to D; therefore the rectangle contained by A, C is equal to the square of B.



And if the rectangle contained by A, C be equal to the square of B; A is to B, as B to C.

The same construction being made, because the rectangle contained by A, C is equal to the square of B, and the square of B is equal to the rectangle contained by B, D, because B is equal to D; there-

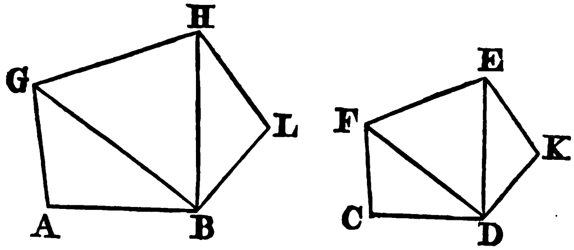
fore the rectangle contained by A, C is equal to that contained by B, D: but if the rectangle contained by the extremes be equal to that contained by the means, the four straight lines are proportionals (16. 6.); therefore A is to B, as D to C, but B is equal to D: wherefore as A to B, so B to C. Therefore if three straight lines, &c. Q. E. D.

PROP. XVIII. PROB.

Upon a given straight line to describe a rectilineal figure similar, and similarly situated to a given rectilineal figure.

Let AB be the given straight line, and CDEF the given rectilineal figure of four sides; it is required upon the given straight line AB to describe a rectilineal figure similar and similarly situated to CDEF.

Join DF, and at the points A, B in the straight line AB make (14. 1.) the angle BAG equal to the angle at C, and the angle ABG equal



to the angle CDF; therefore the remaining angle CFD is equal to the remaining angle AGB (5. 2.); wherefore the triangle FCD is equiangular to the triangle GAB: again, at the points G, B in the straight line GB make (14. 1.) the angle BGH equal to the angle DFE, and the angle GBH equal to FDE; therefore the remaining angle FED is equal to the remaining angle GHB, and the triangle FDE equiangular to the triangle GBH; then, because the angle AGB is equal to the angle CFD, and BGH to DFE, the whole angle AGH is equal to the whole CFE; for the same reason, the angle ABH is equal to the angle CDE; also the angle at A is equal to the angle at C, and the angle GHB to FED: therefore the rectilineal figure ABHG is equiangular to CDEF; but likewise these figures have their sides about the equal angles proportionals; because the triangles GAB, FCD being equiangular, BA is (5. 5.) to AG, as DC to CF; and because AG is to GB, as CF to FD; and as GB to GH, so, by reason of the equiangular triangles BGH, DFE, is FD to FE; therefore, ex æquali (16. 4.), AG is to GH, as CF to FE; in the same manner it may be proved that AB is to BH, as CD to DE; and GH is to HB, as FE to ED (5. Def. 5.). Wherefore, because the rectilineal figures ABHG, CDEF are equiangular, and have their sides about the equal angles proportionals, they are similar to one another.

Next, Let it be required to describe upon a given straight line AB, a rectilineal figure similar, and similarly situated to the rectilineal figure CDKEF of five sides.

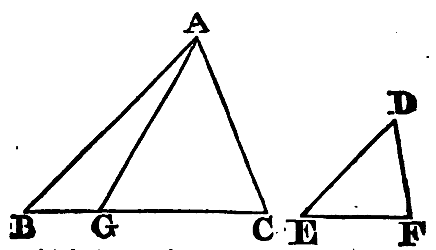
Join DE, and upon the given straight line AB describe the rectilineal figure ABHG similar and similarly situated to the quadrilateral figure CDEF, by the former case; and at the points B, H in the straight line BH, make the angle HBL equal to the angle EDK, and the angle BHL equal to the angle DEK; therefore the remaining angle at K is equal to the remaining angle at L; and because the figures ABHG, CDEF are similar, the angle GHB is equal to the angle FED, and BHL is equal to DEK; wherefore the whole angle GHL is equal to the whole angle FEK; for the same reason, the angle ABL is equal to the angle CDK: therefore the five sided figures AGHLB, CFEKD are equiangular; and because the figures AGHB, CFED are similar, GH is to HB, as FE to ED; and as HB to HL, so is ED to EK (5. 5.); therefore ex æquali (16. 4.), GH is to HL, as FE to EK; for the same reason, AB is to BL, as CD to DK; and BL is to LH, as (5. 5.) DK to KE, because the triangles BLH, DKE are equiangular: therefore because the five sided figures AGHLB, CFEKD are equiangular, and have their sides about the equal angles proportionals, they are similar to one another: and in the same manner a rectilineal figure of six sides may be described upon a given straight line similar to one given, and so on. Which was to be done.

PROP. XIX. THEOR.

Similar triangles are to one another in the duplicate ratio of their homologous sides.

Let ABC, DEF be similar triangles having the angle B equal to the angle E, and let AB be to BC, as DE to EF, so that the side BC is homologous to EF (Def. 10. 5.); the triangle ABC has to the triangle DEF, the duplicate ratio of that which BC has to EF.

Take BG a third proportional to BC, EF (11. 5.), so that BC is to EF, as EF to BG, and join GA: then, because as AB to BC, so DE to EF; alternately (6. 4.), AB is to DE, as BC to EF; but as BC to EF, so is EF to BG; therefore (15. 4.), as AB to DE, so is EF to BG; wherefore the sides of the triangles ABG, DEF which are about the equal angles, are reciprocally



proportional: but triangles which have the sides about two equal angles reciprocally proportional, are equal to one another (15. 5.):

therefore the triangle ABG is equal to the triangle DEF: and because as BC is to EF, so EF to BG; and that if three straight lines be proportionals, the first is said (15. Def. 4.) to have to the third the duplicate ratio of that which it has to the second; BC therefore has to BG the duplicate ratio of that which BC has to EF: but as BC to BG, so is (2. Cor. 2. 5.) the triangle ABC to the triangle ABG. Therefore the triangle ABC has to the triangle ABG the duplicate ratio of that which BC has to EF; but the triangle ABG is equal to the triangle DEF; wherefore also the triangle ABC has to the triangle DEF the duplicate ratio of that which BC has to EF. Therefore, similar triangles, &c. Q. E. D.

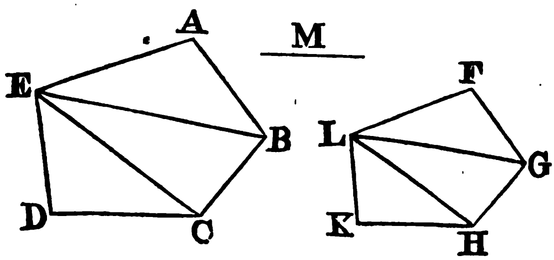
Cor. From this it is manifest, that if three straight lines be proportionals, as the first is to the third, so is any triangle upon the first to a similar and similarly described triangle upon the second.

PROP. XX. THEOR.

Similar polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have: and the polygons have to one another the duplicate ratio of that which their homologous sides have.

Let ABCDE, FGHKL be similar polygons, and let AB be the homologous side to FG: the polygons ABCDE, FGHKL may be divided into the same number of similar triangles, whereof each to each has the same ratio which the polygons have; and the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the side AB has to the side FG.

Join BE, EC, GL, LH: and because the polygon ABCDE is similar to the polygon FGHKL, the angle BAE is equal to the angle

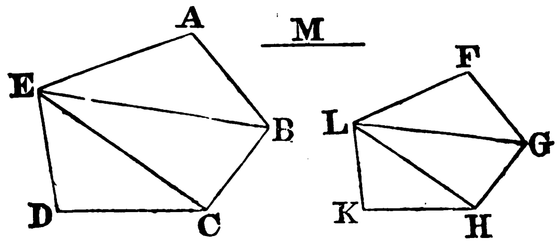


GFL (5. Def. 5.), and BA is to AE, as GF to FL (5. Def. 5.); wherefore because the triangles ABE, FGL, have an angle in one equal to an angle in the other, and their sides about these equal angles proportionals; the triangle ABE is equiangular (7. 5.), and therefore similar to the triangle FGL (5. 5.); wherefore the angle ABE is equal to the angle FGL: and, because the polygons are similar, the

whole angle ABC is equal (5. Def. 5.) to the whole angle FGH; therefore the remaining angle EBC is equal to the remaining angle LGH: and because the triangles ABE, FGL are similar, EB is to BA, as LG to GF (5. Def. 5.); and also, because the polygons are similar, AB is to BC, as FG to GH (5. Def. 5.); therefore, ex æquali (16. 4.), EB is to BC, as LG to GH; that is, the sides about the equal angles EBC, LGH are proportionals; therefore (7. 5.) the triangle EBC is equiangular to the triangle LGH, and similar to it (5. 5.). For the same reason, the triangle ECD likewise is similar to the triangle LKH: therefore the similar polygons ABCDE, FGHKL are divided into the same number of similar triangles.

Also these triangles have, each to each, the same ratio which the polygons have to one another, the antecedents being ABE, EBC, ECD, and the consequents FGL, LGH, LHK: and the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the side AB has to the homologous side FG.

Because the triangle ABE is similar to the triangle FGL, ABE has to FGL the duplicate ratio (19. 5.) of that which the side BE has to the side GL; for the same reason, the triangle BEC has to GLH the duplicate ratio of that which BE has to GL; therefore, as the triangle ABE to the triangle FGL, so (15. 4.) is the triangle BEC to the triangle GLH. Again, because the triangle EBC is similar to the triangle LGH, EBC has to LGH the duplicate ratio of that which the side EC has to the side LH: for the same reason, the triangle ECD has to the triangle LHK, the duplicate ratio of that which EC has to LH; as therefore the triangle EBC to the triangle LGH, so is (15. 4.) the triangle ECD to the triangle LHK; but it has been proved that the triangle EBC is likewise to the triangle LGH, as the triangle ABE to the triangle FGL. Therefore, as the triangle ABE is to the triangle FGL, so is the triangle EBC to the triangle



LGH, and the triangle ECD to the triangle LHK; and therefore as one of the antecedents to one of the consequents, so are all the antecedents to all the consequents. Wherefore as the triangle ABE to the triangle FGL, so is the polygon ABCDE to the polygon FGHKL; but the triangle ABE has to the triangle FGL, the duplicate ratio of that which the side AB has to the homologous side FG. Therefore also the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which AB has to the homologous side FG. Wherefore similar polygons, &c. Q. E. D.

COR. 1. In like manner it may be proved that similar four sided figures, or of any number of sides are one to another in the duplicate ratio of their homologous sides, and it has already been proved in triangles. Therefore universally, similar rectilineal figures are to one another in the duplicate ratio of their homologous sides.

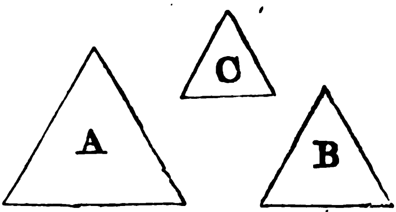
COR. 2. And if to AB, FG two of the homologous sides a third proportional M be taken, AB has (15. Def. 4.) to M the duplicate ratio of that which AB has to FG; but the four sided figure or polygon upon AB has to the four sided figure or polygon upon FG likewise the duplicate ratio of that which AB has to FG: therefore as AB is to M, so is the figure upon AB to the figure upon FG, which was also proved in triangles (Cor. 19. 5.). Therefore, universally, it is manifest, that if three straight lines be proportionals, as the first is to the third, so is any rectilineal figure upon the first, to a similar and similarly described rectilineal figure upon the second.

PROP. XXI. THEOR.

Rectilineal figures which are similar to the same rectilineal figure, are also similar to one another.

Let each of the rectilineal figures, A, B be similar to the rectilineal figure C; the figure A is similar to the figure B.

Because A is similar to C; they are equiangular, and also have their sides about the equal angles proportionals (5. Def. 5.). Again,



because B is similar to C, they are equiangular, and have their sides about the equal angles proportionals (5. Def. 5.): therefore the figures A, B are each of them equiangular to C, and have the sides about the equal angles of each of them and of C proportionals. Wherefore the rectilineal figures A and B are equiangular, and have their sides about the equal angles proportionals (15. 4.). Therefore A is similar (5. Def. 5.) to B. Q. E. D.

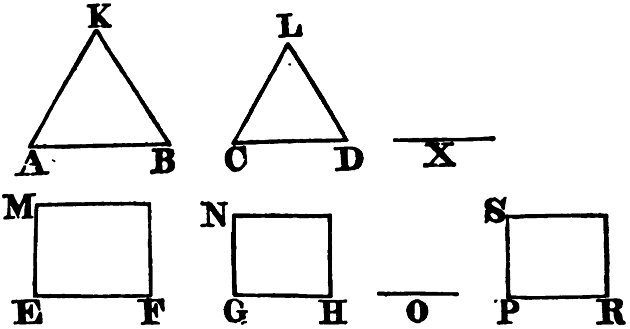
PROP. XXII. THEOR.

If four straight lines be proportionals, the similar rectilineal figures similarly described upon them shall also be proportionals; and if the similar rectilineal figures similarly described upon four straight lines be proportionals, those straight lines shall be proportionals.

Let the four straight lines AB, CD, EF, GH be proportionals, viz.

AB to CD, as EF to GH, and upon AB, CD let the similar rectilineal figures KAB, LCD be similarly described; and upon EF, GH the similar rectilineal figures MF, NH in like manner: the rectilineal figure KAB is to LCD, as MF to NH.

To AB, CD take a third proportional (11. 5.) X; and to EF, GH a third proportional O: and because AB is to CD, as EF to GH, and



that CD is (15. 4.) to X, as GH to O; wherefore, ex æquali (16. 4.), as AB to X, so is EF to O: but as AB to X, so is (2. Cor. 20. 5.) the rectilineal KAB to the rectilineal LCD, and as EF to O, so is (2. Cor. 20. 5.) the rectilineal MF to the rectilineal NH: therefore as KAB to LCD, so (15. 4.) is MF to NH.

And if the rectilineal KAB be to LCD, as MF to NH; the straight line AB is to CD, as EF to GH.

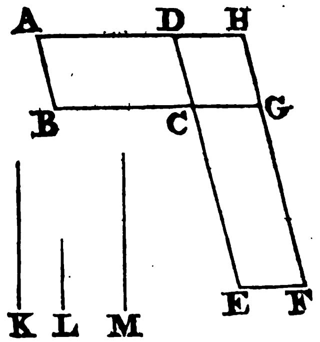
Make (12. 5.) as AB to CD, so EF to PR, and upon PR describe (18. 5.) the rectilineal figure SR similar and similarly situated to either of the figures MF, NH: then because as AB to CD, so EF to PR, and that upon AB, CD are described the similar and similarly situated rectilineals KAB, LCD, and upon EF, PR, in like manner, the similar rectilineals MF, SR; KAB is to LCD, as MF to SR; but, by the Hypothesis, KAB is to LCD, as MF to NH; and therefore the rectilineal MF having the same ratio to each of the two NH, SR, these are equal (14. 4.) to one another; they are also similar, and similarly situated; therefore GH is equal to PR: and because as AB to CD, so is EF to PR, and that PR is equal to GH; AB is to CD, as EF to GH. If therefore four straight lines, &c. Q. E. D.

PROP. XXIII. THEOR.

Equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides.

Let AC, CF be equiangular parallelograms, having the angle BCD equal to the angle ECG; the ratio of the parallelogram AC to the parallelogram CF, is the same with the ratio which is compounded of the ratios of their sides.

Let BC, CG, be placed in a straight line; therefore DC and CE are also in a straight line (2. 1.); and complete the parallelogram DG; and, taking any straight line K, make (12. 5.) as BC to CG, so K to L: and as DC to CE, so make (12. 5.) L to M: therefore the ratios of K to L, and L to M, are the same with the ratios of the sides, viz. of BC to CG, and DC to CE. But the ratio of K to M is that which is said to be compounded (15. Def. 4.) of the ratios of K to L, and L to M; Wherefore also K has to M the ratio compounded of the ratios of the sides; and because as BC to CG, so is the parallelogram AC to the parallelogram CH (2. 5.); but as BC to CG, so is K to L; therefore K is (15. 4.) to L, as the parallelogram AC to the parallelogram GH: again, because as DC to CE, so is the parallelogram CH to the parallelogram CF; but as DC to CE, so is L to M; wherefore L is (15. 4.) to M, as the parallelogram CH to the parallelogram CF: therefore, since it has been proved, that as K to L, so is the parallelogram AC to the parallelogram CH; and as L to M, so the parallelogram CH to the parallelogram CF; (16. 4.) K is to M, as the parallelogram AC to the parallelogram CF: but K has to M the ratio which is compounded of the ratios of the sides: therefore also the parallelogram AC has to the parallelogram CF the ratio which is compounded of the ratios of the sides. Wherefore, equiangular parallelograms, &c. Q. E. D.

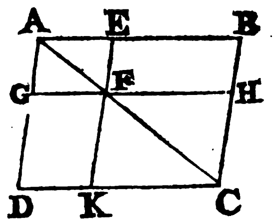


PROP. XXIV. THEOR.

The parallelograms about the diameter of any parallelogram are similar to the whole, and to one another.

Let ABCD be a parallelogram, of which the diameter is AC; and EG, HK the parallelograms about the diameter: the parallelograms EG, HK are similar both to the whole parallelogram ABCD, and to one another.

Because DC, GF are parallels, the angle ADC is equal (3. 2.) to the angle AGF: for the same reason, because BC, EF are parallels, the angle ABC is equal to the angle AEF; and each of the angles BCD, EFG is equal to the opposite angle DAB (9. 2.), and therefore are equal to one another; wherefore the parallelograms ABCD, AEFG are equiangular; and because the angle ABC is equal to the angle AEF, and the angle BAC common to the two triangles BAC, EAF, they are equiangular to one another; therefore (5. Def. 5.) as AB to BC, so is AE to EF: and because the opposite sides of parallelograms are equal to one another (9. 2.), AB is (13. 4.) to AD,

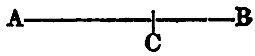


as AE to AG; and DC to CB, as GF to FE; and also CD to DA, as FG to GA: therefore the sides of the parallelograms ABCD, AEFG about the equal angles are proportionals; and they are therefore similar to one another (5. Def. 5.); for the same reason, the parallelogram ABCD is similar to the parallelogram FHCK. Wherefore each of the parallelograms GE, KH is similar to DB: but rectilineal figures which are similar to the same rectilineal figure, are also similar to one another (21. 5.), therefore the parallelogram GE is similar to KH. Wherefore the parallelograms, &c. Q. E. D.

PROP. XXV. PROB.

To cut a given straight line in extreme and mean ratio.

Let AB be the given straight line; it is required to cut it in extreme and mean ratio.



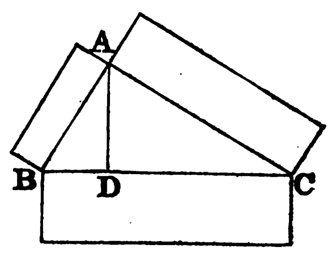
Divide AB in the point C, so that the rectangle contained by AB, BC be equal to the square of AC (24. 2.); then because the rectangle AB, BC is equal to the square of AC, as BA to AC, so is AC to CB (17. 5.): therefore AB is cut in extreme and mean ratio in C (2. Def. 5.). Which was to be done.

PROP. XXVI. THEOR.

In right angled triangles the rectilineal figure described upon the side opposite to the right angle, is equal to the similar, and similarly described figures upon the sides containing the right angle.

Let ABC be a right angled triangle, having the right angle BAC; the rectilineal figure described upon BC is equal to the similar and similarly described figures upon BA, AC.

Draw the perpendicular AD; therefore because in the right angled triangle ABC, AD is drawn from the right angle at A perpendicular to the base BC, the triangles ABD, ADC are similar to the whole triangle ABC, and to one another: and because the triangle ABC is similar to ADB, as CB to BA, so is BA to BD (5. 5.); and because these three straight lines are proportionals, as the first to the third, so is the figure upon the



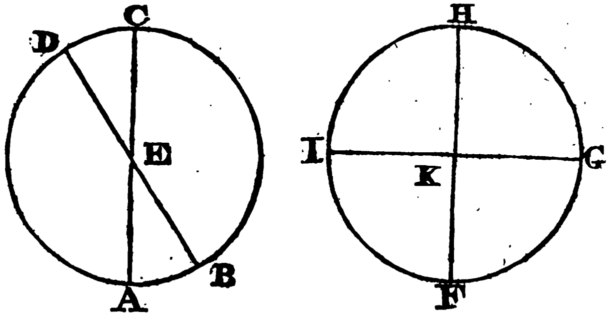
first to the similar, and similarly described figure upon the second (2. Cor. 20. 5.): therefore as CB to BD, so is the figure upon CD to the similar and similarly described figure upon BA: and, inversely (5. 4.), as DB to BC, so is the figure upon BA to that upon BC; for the same reason, as DC to CB, so is the figure upon CA to that upon CB. Wherefore as BD and DC together to BC, so are the figures upon BA, AC to that upon BC (18. 4.): but BD and DC together are equal to BC; therefore the figure described on BC is equal to the similar and similarly described figures on BA, AC. Wherefore, in right angled triangles, &c. Q. E. D.

PROP. XXVII. THEOR.

In equal circles, angles at the centres have the same ratio to one another as the circumferences on which they stand.

Let ABCD and FGHI be equal circles, and let the angles AEB and FKG be at the centre; they have the same ratio to each other as the circumferences on which they stand, AB and FG.

Produce AE, BE to the circumference at D and C; and FK, GK to the circumference at H and I; then the angles at E are together



equal to four right angles (1. Cor. 3. 1.), and they stand on the whole circumference; and therefore, whatever part the angle AEB is of four right angles, the arc AB is of the whole circumference; and therefore, as the angle AEB is to four right angles, so is the arc AB to the whole circumference ABCD (18. Def. 4.). It may be shewn in the same way, that as the angle FKG is to four right angles, so is the arc FG to the whole circumference FGHI; but the whole circumference FGHI is equal to the whole circumference ABCD by the hypothesis: and therefore, as the angle AEB is to the angle FKG, so is the arc AB to the arc FG. Therefore, in equal circles, &c. Q. E. D.

Cor. I. It may be shewn in the same way, that in equal circles the sectors have the same ratio to each other as the circumferences on which they stand.

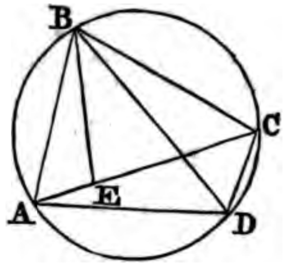
Cor. II. Because the angle at the centre is double the angle at the circumference upon the same base; therefore, in equal circles angles at the circumference have the same ratio to one another as the circumferences on which they stand.

PROP. XXVIII. THEOR.

The rectangle contained by the diagonals of a quadrilateral inscribed in a circle, is equal to both the rectangles contained by its opposite sides.

Let ABCD be any quadrilateral inscribed in a circle, and join AC, BD; the rectangle contained by AC, BD is equal to the two rectangles contained by AB, CD and by AD, BC.

Make the angle ABE equal to the angle DBC; add to each of these the common angle EBD, then the angle ABD is equal to the angle EBC; and the angle BDA is equal (13. 3.) to the angle BCE, because they are in the same segment; therefore the triangle ABD is equiangular to the triangle BCE: wherefore (5. 5.) as BC is to CE, so is BD to DA, and consequently the rectangle BC, AD is equal (16. 5.) to the rectangle BD, CE: again, because the angle ABE is equal to the angle DBC, and the angle (13. 3.) BAE to the angle BDC, the triangle ABE is equiangular to the triangle BCD: as therefore BA to AE, so is BD to DC; wherefore the rectangle BA, DC is equal to the rectangle BD, AE: but the rectangle BC, AD has been shewn equal to the rectangle BD, CE; therefore the whole rectangle AC, BD is equal to the rectangle AB, DC together with the rectangle AD, BC. Therefore, the rectangle, &c. Q. E. D.



THE
ELEMENT
OF
GEOMETRY.

BOOK VI.

DEFINITIONS.

I.

When all the angles of a plane rectilineal figure are upon the circumference of a circle, the figure may be said to be inscribed in the circle, or the circle circumscribed about the figure.

II.

When each side of a figure touches the circumference of a circle, the figure may be said to be circumscribed, or the circle inscribed.

III.

A plane rectilineal equilateral and equiangular figure may be called a regular polygon.

IV.

The sum of the sides of a figure may be called the perimeter of the figure.

V.

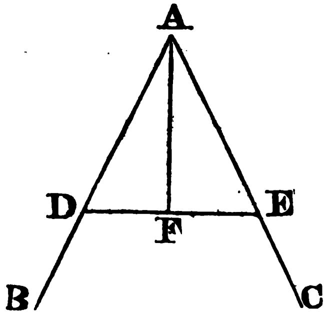
When the extremities of a straight line are in the circumference of a circle, the line may be said to be placed in the circle.

PROPOSITION I. PROBLEM.

To bisect a given angle.

Let BAC be the given angle, it is required to bisect it.

In AB take any point D; from A, with the radius AD, describe an arc intersecting AC in E; join DE, bisect it in F (8. 1.), and join AF: the angle DAF is equal to the angle EAF.

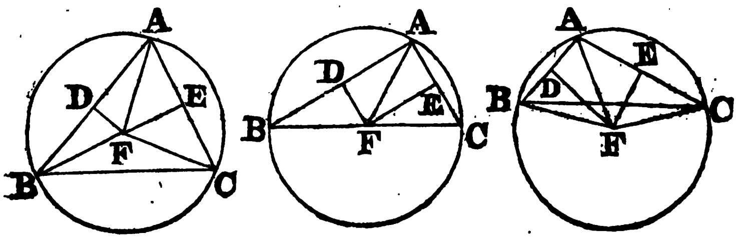


Because the triangles DAF and EAF have the side DA equal to EA (Def. 54. 1.), DF equal to EF, by the construction, and FA common; the three sides of the one are equal to the three sides of the other, and therefore the triangles are equivalent (7. 1.); and the angle DAF is equal to the angle EAF; and therefore the straight line FA bisects the angle DAE, or BAC. Which was to be done.

PROP II. PROB.

To describe a circle about a given triangle.

Let the given triangle be ABC; it is required to describe a circle about ABC.



Bisect (8. 1.) AB, AC in the points D, E, and from these points draw DF, EF at right angles (9. 1.) to AB, AC; DF, EF produced meet one another: for, if they do not meet, they are parallel, wherefore AB, AC, which are at right angles to them, are parallel; which is absurd: let them meet in F, and join FA; also, if the point F be not in BC, join BF, CF: then, because AD is equal to DB, and DF common, and at right angles to AB, the base AF is equal (4. 1.) to the base FB: in like manner, it may be shown, that CF is equal to FA; and therefore BF is equal to FC; and FA, FB, FC are equal to one another: wherefore the circle described from the centre F, at the distance of one of them, shall pass through the extremities of the other two, and be described about the triangle ABC. Which was to be done.

Cor. I. In the same way the circumference of a circle may be described through any three points, not in a straight line.

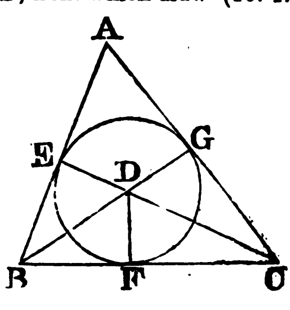
Cor. II. If a segment of a circle is given, the circle may be described of which it is a segment.

PROP. III. PROB.

To inscribe a circle in a given triangle.

Let the given triangle be ABC; it is required to inscribe a circle in ABC.

Bisect (1. 6.) the angles ABC, BCA by the straight lines BD, CD meeting one another in the point D, from which draw (10. 1.) DE, DF, DG perpendiculars to AB, BC, CA; and because the angle EBD is equal to the angle FBD, for the angle ABC is bisected by BD, and that the right angle BED is equal to the right angle BFD, the two triangles EBD, FBD have two angles of the one equal to two angles of the other, and the side BD, which is opposite to one of the equal angles in each, is common to both; therefore their other sides shall be equal (6. 2.); wherefore DE is equal to DF: for the same reason, DG is equal to DF; therefore the three straight lines DE, DF, DG are equal to one another, and the circle described from the centre D, at the distance of any of them, shall pass through the extremities of the other two, and touch the straight lines AB, BC, CA, because the angles at the points E, F, G are right angles, and the straight line which is drawn from the extremity of a diameter at right angles to it, touches (8. 3.) the circle; therefore the straight lines AB, BC, CA do each of them touch the circle, and the circle EFG is inscribed in the triangle ABC. Which was to be done.

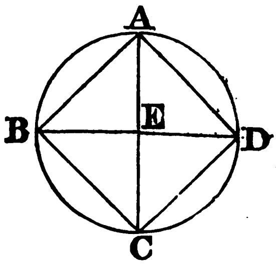


PROP. IV. PROB.

To inscribe a square in a given circle.

Let ABCD be the given circle; it is required to inscribe a square in ABCD.

Draw the diameters AC, BD at right angles to one another; and join AB, BC, CD DA.; because BE is equal to ED, for E is the centre, and that EA is common, and at right angles to BD; the base BA is equal (4. 1.) to the base AD; and for the same reason, BC, CD are each of them equal to BA or AD; therefore the quadrilateral figure ABCD is equilateral. It is also rectangular; for the straight line BD, being the diameter of the circle ABCD, BAD is a semicircle; wherefore the angle BAD is a right (22. 3.) angle; for the same reason each of the angles ABC, BCD, CDA is a right angle; therefore the quadrilateral figure ABCD is rectangular, and it has been shown to be equilateral; therefore it is a square; and it is inscribed in the circle ABCD. Which was to be done.

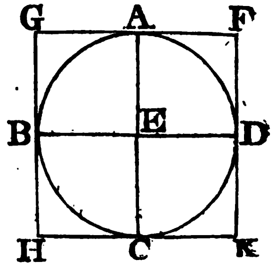


PROP. V. PROB.

To describe a square about a given circle.

Let ABCD be the given circle; it is required to describe a square about it.

Draw two diameters AC, BD of the circle ABCD, at right angles to one another, and through the points A, B, C, D draw (9. 3.) FG, GH, HK, KF touching the circle; and because FG touches the circle ABCD, and EA is drawn from the centre E to the point of contact A, the angles at A are right (10. 3.) angles: for the same reason, the angles at the points B, C, D are right angles; and because the angle AEB is a right angle, as likewise is EBG, GH is parallel (2. Cor. 2. 2.) to AC: for the same reason, AC is parallel to FK; and in like manner GF, HK may each of them be demonstrated to be pa-



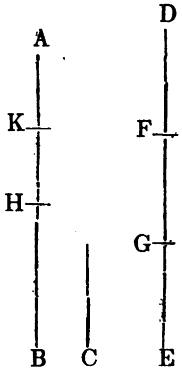
rallel to BED; therefore the figures GK, GC, AK, FB, BK are parallelograms, and GF is therefore equal (9. 2.) to HK, and GH to FK: and because AC is equal to BD, and that AC is equal to each of the two GH, FK; and BD to each of the two GF, HK; GH, FK are each of them equal to GF or HK: therefore the quadrilateral figure FGHK is equilateral. It is also rectangular; for GBEA being a parallelogram, and AEB a right angle, AGB is (9. 2.) likewise a right angle: in the same manner it may be shewn that the angles at H, K, F are right angles: therefore the quadrilateral figure FGHK is rectangular; and it was demonstrated to be equilateral; therefore it is a square; and it is described about the circle ABCD. Which was to be done.

PROP. VI. THEOR.

If from the greater of two unequal magnitudes, there be taken more than its half, and from the remainder more than its half, and so on: there shall at length remain a magnitude less than the least of the proposed magnitudes.

Let AB and C be two unequal magnitudes, of which AB is the greater. If from AB there be taken more than its half, and from the remainder more than its half, and so on; there shall at length remain a magnitude less than C.

For C may be multiplied, so as at length to become greater than AB. Let it be so multiplied, and let DE its multiple be greater than AB, and let DE be divided into DF, FG, GE, each equal to C. From AB take BH greater than its half, and from the remainder AH take HK greater than its half, and so on, until there be as many divisions in AB as there are in DE: and let the divisions in AB be AK, KH, HB; and the divisions in ED be DF, FG, GE. And because DE is greater than AB, and that EG taken from DE is not greater than its half, but BH taken from AB is greater than its half; therefore the remainder GD is greater than the remainder HA. Again, because GD is greater than HA, and that GF is not greater than the half of GD, but HK is greater than the half of HA; therefore the remainder FD is greater than the remainder AK. And FD is equal to C, therefore C is greater than AK; that is, AK is less than C. Q. E. D.



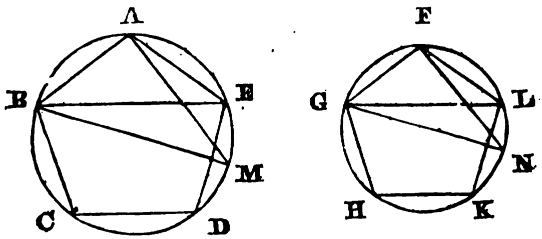
And if only the halves be taken away, the same thing may in the same way be demonstrated.

PROP. VII. THEOR.

Similar polygons inscribed in circles are to one another as the squares of their diameters.

Let ABCDE, FGHKL be two circles, and in them the similar polygons ABCDE, FGHKL; and let BM, GN be the diameters of the circles; as the square of BM is to the square of GN, so is the polygon ABCDE to the polygon FGHKL.

Join BE, AM, GL, FN: and because the polygon ABCDE is similar to the polygon FGHKL, and similar polygons are divided into similar triangles (20. 5.); the triangles ABE, FGL are similar and



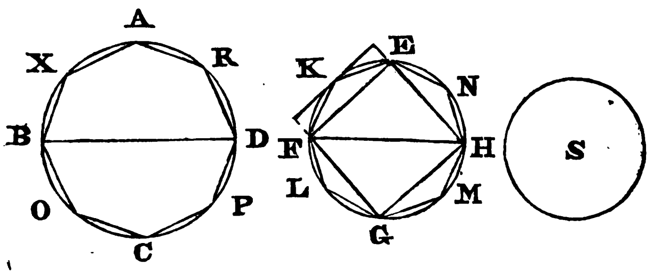
equiangular (7. 5.); and therefore the angle AEB is equal to the angle FLG: but AEB is equal (13. 3.) to AMB, because they stand upon the same circumference; and the angle FLG is, for the same reason, equal to the angle FNG: therefore also the angle AMB is equal to FNG: and the right angle BAM is equal to the right (22. 3.) angle GFN; wherefore the remaining angles in the triangles ABM, FGN are equal, and they are equiangular to one another; and therefore as BM to GN so (5. 5.) is BA to GF, and therefore the duplicate ratio of BM to GN, is the same (16. 4.) with the duplicate ratio of BA to GF: but the ratio of the square of BM to the square of GN, is the duplicate (20. 5.) ratio of that which BM has to GN; and the ratio of the polygon ABCDE to the polygon FGHKL, is the duplicate ratio of that which BA has to GF; therefore, as the square of BM to the square of GN, so is the polygon ABCDE to the polygon FGHKL. Wherefore, similar polygons, &c. Q. E. D.

PROP. VIII. THEOR.

Circles are to one another as the squares of their diameters.

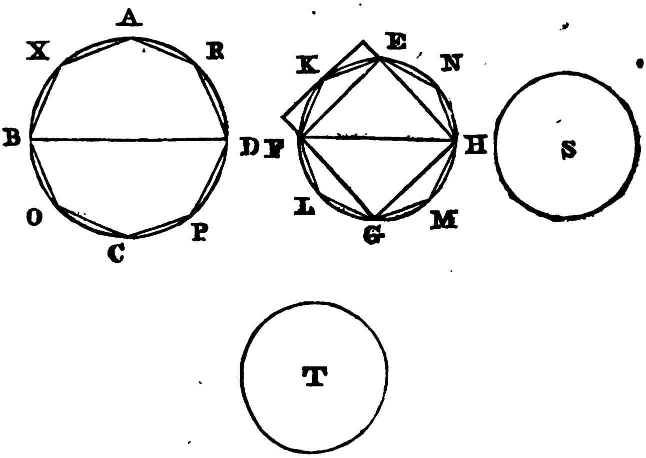
Let ABCD, EFGH be two circles, and BD, FH their diameters; as the square of BD to the square of FH, so is the circle ABCD to the circle EFGH.

For, if it be not so, the square of BD shall be to the square of FH, as the circle ABCD is to some space either less than the circle EFGH, or greater than it. First, let it be to a space S less than the circle EFGH; and in the circle EFGH describe the square EFGH; this square is greater than half of the circle EFGH; because, if through the points E, F, G, H, there be drawn tangents to the circle, the square EFGH is half (15. 2.) of the square described about the circle; and the circle is less than the square described about it; therefore the square EFGH is greater than half of the circle. Divide the circumferences EF, FG, GH, HE, each into two equal parts in the points K, L, M, N, and join EK, KF, FL, LG, GM, MH, HN, NE; therefore each of the triangles EKF, FLG, GMH, HNE is greater than half of the segment of the circle it stands in; because if straight lines touching the circle be drawn through the points K, L, M, N, and parallelograms upon the straight lines EF, FG, GH, HE, be completed; each of the triangles EKF, FLG, GMH, HNE shall be the half (15. 2.) of the parallelogram in which it is; but every segment is less than the parallelogram in which it is; wherefore each of the triangles EKF, FLG, GMH, HNE is greater than half the segment of the circle which contains it: and if these circumferences before named be divided each into two equal parts, and their extremities be joined by straight lines, by continuing to do this, there



will at length remain segments of the circle which together shall be less than the excess of the circle EFGH above the space S: because (6. 6.), if from the greater of two unequal magnitudes there be taken more than its half, and from the remainder more than its half, and so on, there shall at length remain a magnitude less than the least of the proposed magnitudes. Let then the segments EK, KF, FL, LG, GM, MH, HN, NE be those that remain and are together less than the excess of the circle EFGH above S: therefore the rest of the circle, viz. the polygon EKFLGMHN is greater than the space S. Describe likewise in the circle ABCD the polygon AXBOCPDR similar to the polygon EKFLGMHN: as, therefore, the square of BD is to the square of FH, so (5. 6.) is the polygon AXBOCPDR to the polygon EKFLGMHN: but the square of BD is also to the square of FH, as the circle ABCD is to the space S: therefore, as the circle ABCD is to the space S, so is (15. 4.) the

polygon AXBOCPDR to the polygon EKFLGMHN : but the circle ABCD is greater than the polygon contained in it: wherefore the space S is greater (20. Def. 4.) than the polygon EKFLGMHN: but it is likewise less, as has been demonstrated : which is impossible. Therefore the square of BD is not to the square of FH, as the circle ABCD is to any space less than the circle EFGH. In the same manner, it may be demonstrated, that neither is the square of FH to the square of BD, as the circle EFGH is to any space less than the circle ABCD. Nor is the square of BD to the square of FH, as the circle ABCD is to any space greater than the circle EFGH: for, if possible, let it be so to T, a space greater than the circle EFGH : therefore, inversely, as the square of FH to the square

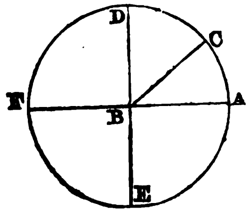


of BD, so is the space T to the circle ABCD. But as the space T is to the circle ABCD, so is the circle EFGH to some space, which must be less (20. Def. 4.) than the circle ABCD, because the space T is greater, by hypothesis, than the circle EFGH. Therefore, as the square of FH is to the square of BD, so is the circle EFGH to a space less than the circle ABCD, which has been demonstrated to be impossible: therefore the square of BD is not to the square of FH, as the circle ABCD is to any space greater than the circle EFGH : and it has been demonstrated, that neither is the square of BD to the square of FH, as the circle ABCD to any space less than the circle EFGH : wherefore, as the square of BD to the square of FH, so is the circle ABCD to the circle EFGH. Circles therefore are, &c. Q. E. D.

PLANE TRIGONOMETRY.

LEMMA I.

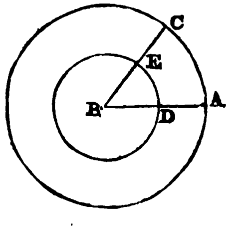
LET ABC be a rectilineal angle; if about the point B as a centre, and with any distance BA, a circle be described, meeting BA, BC,



the straight lines including the angle ABC in A, C; the angle ABC will be to four right angles, as the arc AC to the whole circumference. (20. Def. 4.).

LEMMA II.

Let ABC be a plane rectilineal angle as before: about B as a centre, with any two distances BD, BA, let two circles be described meeting BA, BC in D, E, A, C; the arc AC will be to the whole circumference of which it is an arc, as the arc DE is to the whole circumference of which it is an arc.



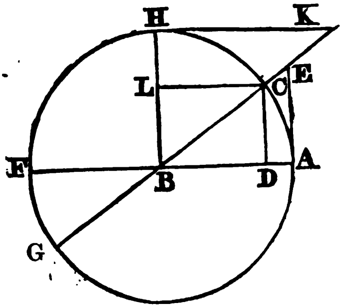
By Lemma I. the arc AC is to the whole circumference of which it is an arc, as the angle ABC is to four right angles; and by the

same Lemma I. the arc DE is to the whole circumference of which it is an arc, as the angle ABC is to four right angles; therefore the arc AC is to the whole circumference of which it is an arc, as the arc DE to the whole circumference of which it is an arc, (15. 4.).

DEFINITIONS.

I.

LET ABC be a plane rectilineal angle; if about B as a centre, with BA any distance, a circle ACF be described, meeting BA, BC in A, C; the arc AC may be called the measure of the angle ABC.



II.

The circumference of a circle may be supposed to be divided into 360 equal parts called degrees; and each degree into 60 equal parts called minutes, and each minute into 60 equal parts called seconds, &c. And as many degrees, minutes, seconds, &c. as are contained in any arc, of so many degrees, minutes, seconds, &c. is the angle, of which that arc is the measure, said to be.

COR. Whatever be the radius of the circle of which the measure of a given angle is an arc, that arc will contain the same number of degrees, minutes, seconds, &c. as is manifest from Lemma II.

III.

Let AB be produced till it meet the circle again in F, the angle CBF, which, together with ABC, is equal to two right angles, may be called the *Supplement* of the angle ABC.

IV.

A straight line CD drawn through C, one of the extremities of the arc AC, perpendicular upon the diameter passing through the other extremity A, may be called the *Sine* of the arc AC, or of the angle ABC, of which it is the measure.

COR. The *Sine* of a quadrant, or of a right angle, is equal to the radius.

V.

The segment DA of the diameter passing through A, one extremity of the arc AC, between the sine CD and that extremity, may be called the *Versed Sine* of the arc AC, or angle ABC.

VI.

A straight line AE touching the circle at A, one extremity of the arc AC, and meeting the diameter BC passing through the other extremity C in E, may be called the *Tangent* of the arc AC, or of the angle ABC.

VII.

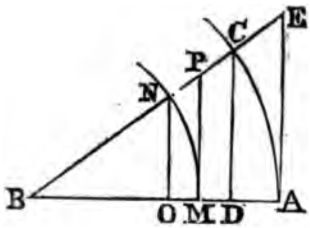
The straight line BE between the centre and the extremity of the tangent AE, may be called the *Secant* of the arc AC, or angle ABC.

Cor. to Def. 4. 6. 7. The sine, tangent, and secant of any angle ABC, are likewise the sine, tangent, and secant of its supplement CBF.

It is manifest from Def. 4. that CD is the sine of the angle CBF. Let CB be produced till it meet the circle again in G; and it is manifest that AE is the tangent, and BE the secant, of the angle ABG or EBF, from Def. 6. 7.

Cor. to Def. 4. 5. 6. 7. The sine, versed sine, tangent, and secant, of any arc which is the measure of any given angle ABC, is to the sine, versed sine, tangent, and secant, of any other arc which is the measure of the same angle, as the radius of the first is to the radius of the second.

Let AC, MN be measures of the angles ABC, according to def. 1. CD the sine, DA the versed sine, AE the tangent, and BE the

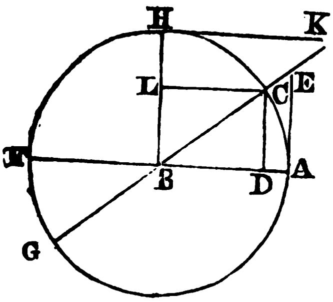


secant of the arc AC, according to def. 4. 5. 6. 7., and NO the sine, OM the versed sine, MP the tangent, and BP the secant of the arc MN, according to the same definitions. Since CD, NO, AE, MP are parallel, CD is to NO as the radius CB to the radius NB, and AE to MP as AB to BM, and BC or BA to BD as BN or BM to BO; and (8. 4.), DA to MO as AB to MB. Hence the corollary is manifest; therefore, if the radius be supposed to be divided into any given number of equal parts, the sine, versed sine, tangent, and secant of any given angle, will each con-

tain a given number of these parts; and, by trigonometrical tables, the length of the sine, versed sine, tangent, and secant of any angle may be found in parts of which the radius contains a given number: and, vice versa, a number expressing the length of the sine, versed sine, tangent, and secant being given, the angle of which it is the sine, versed sine, tangent, and secant, may be found.

VIII.

The difference of an angle from a right angle, may be called the *Complement* of that angle. Thus, if BH be drawn perpendicular



to AB, the angle CBH will be the complement of the angle ABC, or of CBF.

IX.

Let HK be the tangent, CL or DB, which is equal to it, the sine, and BK the secant of CBH, the complement of ABC, according to def. 4. 6. 7, HK is called the *co-tangent*, BD the *co-sine*, and BK the *co-secant* of the angle ABC.

Cor. 1. The radius is a mean proportional between the tangent and co-tangent.

For, since HK, BA are parallel, the angles HKB, ABC will be equal, and the angles KHB, BAE are right; therefore the triangles BAE, KHB are similar, and therefore AE is to AB, as BH or BA to HK.

Cor. 2. The radius is a mean proportional between the co-sine and secant of any angle ABC.

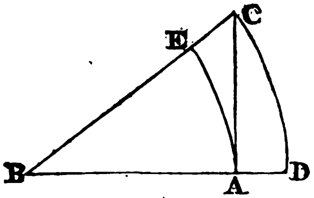
Since CD, AE are parallel, BD is to BC or BA, as BA to BE.

PROP. I.

In a right angled plane triangle, if the hypothenuse be made radius, the sides become the sines of the angles opposite to them: and if either side be made radius, the remaining side is the tangent of the angle opposite to it, and the hypothenuse the secant of the same angle.

Let ABC be a right angled triangle; if the hypothenuse BC be made radius, either of the sides AC will be the sine of the angle ABC opposite to it; and if either side BA be made radius, the other side AC will be the tangent of the angle ABC opposite to it, and the hypothenuse BC the secant of the same angle.

About B as a centre, with BC, BA for distances, let two circles CD, EA be described meeting BA, BC in D, E: since CAB is a



right angle, BC being radius, AC is the sine of the angle ABC by def. 4., and BA being radius, AC is the tangent, and BC the secant of the angle ABC, by def. 6. 7.

Cor. 1. Of the hypothenuse a side and an angle of a right angled triangle, any two being given, the third is also given.

Cor. 2. Of the two sides and an angle of a right angled triangle, any two being given, the third is also given.

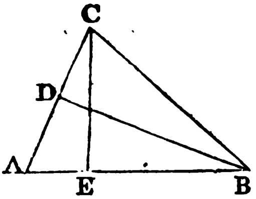
PROP. II.

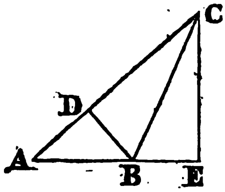
The sides of a plane triangle are to one another as the sines of the angles opposite to them.

In right angled triangles, this Prop. is manifest from Prop. 1. for if the hypothenuse be made radius, the sides are the sines of the angles opposite to them, and the radius is the sine of a right angle (cor. to def. 4.) which is opposite to the hypothenuse.

In any oblique angled triangle ABC, any two sides AB, AC will be to one another as the sines of the angles ACB, ABC which are opposite to them.

From C, B draw CE, BD perpendicular upon the opposite sides AB, AC produced, if need be. Since CEB, CDB are right angles,





BC being radius, CE is the sine of the angle CBA, and BD the sine of the angle ACB: but the two triangles CAE, DAB have each a right angle at D and E; and likewise the common angle CAB; therefore they are similar, and consequently, CA is to AB, as CE to DB; that is, the sides are as the sines of the angles opposite to them.

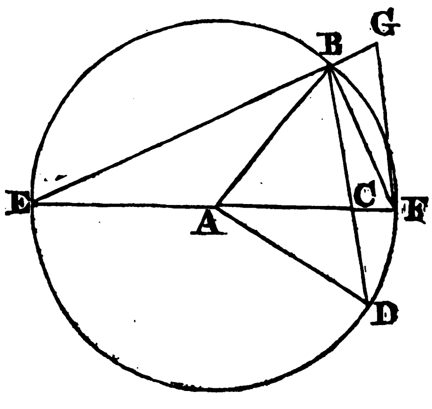
Cor. Hence of two sides, and two angles opposite to them, in a plane triangle, any three being given, the fourth is also given.

PROP. III.

In a plane triangle, the sum of any two sides is to their difference, as the tangent of half the sum of the angles at the base, to the tangent of half their difference.

Let ABC be a plane triangle, the sum of any two sides, AB, AC will be to their difference as the tangent of half the sum of the angles at the base ABC, ACB to the tangent of half their difference.

About A as a centre, with AB the greater side for a distance, let a circle be described, meeting AC produced in E, F, and BC in D; join DA, EB, FB: and draw FG parallel to BC, meeting EB in G.

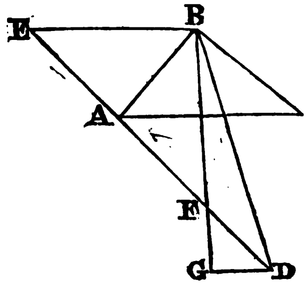


The angle EAB (5. 2.) is equal to the sum of the angles at the base, and the angle EFB at the circumference is equal to the half of EAB at the centre (12. 3.); therefore EFB is half the sum of the angles at the base; but the angle ACB (5. 2.) is equal to the angles CAD and ADC, or ABC together: therefore FAD is the difference of the angles at the base, and FBD at the circumference, or BFG, on account of the parallels FG, BD, is the half of that difference; but since the angle EBF in a semicircle is a right angle, FB being radius, BE, BG, are (1. of this) the tangents of the angles EFB, BFG; but it is manifest that EC is the sum of the sides BA, AC, and CF their difference; and since BC, FG are parallel, (3. 5.) EC is to CF, as EB to BG; that is, the sum of the sides is to their difference, as the tangent of half the sum of the angles at the base to the tangent of half their difference.

PROP. IV.

In any plane triangle BAC, *whose two sides are* BA, AC, *and base* BC, *the less of the two sides, which let be* BA, *is to the greater* AC *as the radius is to the tangent of an angle, and the radius is to the tangent of the excess of this angle above half a right angle as the tangent of half the sum of the angles* B *and* C *at the base, is to the tangent of half their difference.*

At the point A, draw the straight line EAD perpendicular to BA; make AE, AF, each equal to AB, and AD to AC; join BE, BF, BD, and from D, draw DG perpendicular upon BF. And because BA is at right angles to EF, and EA, AB, AF are equal, each of the angles EBA, ABF is half a right angle, and the whole EBF is a right angle; also (4. 1.) EB is equal to BF. And since EBF,



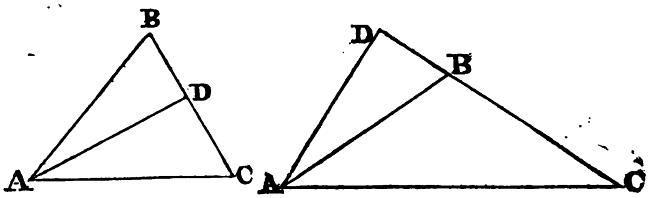
FGD are right angles, EB is parallel to GD, and the triangles EBF, FGD are similar; therefore EB is to BF as DG to GF, and EB being equal to BF, FG must be equal to GD. And because BAD is a right angle, BA the less side is to AD or AC the greater, as the radius is to the tangent of the angle ABD; and because BGD is a right angle, BG is to GD or GF as the radius is to the tangent of

GBD, which is the excess of the angle ABD above ABF half a right angle. But because EB is parallel to GD, BG is to GF as ED is to DF, that is, since ED is the sum of the sides BA, AC and FD their difference, (3. of this,) as the tangent of half the sum of the angles B, C, at the base to the tangent of half their difference. Therefore, in any plane triangle, &c. Q. E. D.

PROP. V.

In any triangle, twice the rectangle contained by any two sides is to the difference of the sum of the squares of these two sides, and the square of the base, as the radius is to the co-sine of the angle included by the two sides.

Let ABC be a plane triangle, twice the rectangle ABC contained by any two sides BA, BC is to the difference of the sum of the squares of BA, BC, and the square of the base AC, as the radius to the co-sine of the angle ABC.

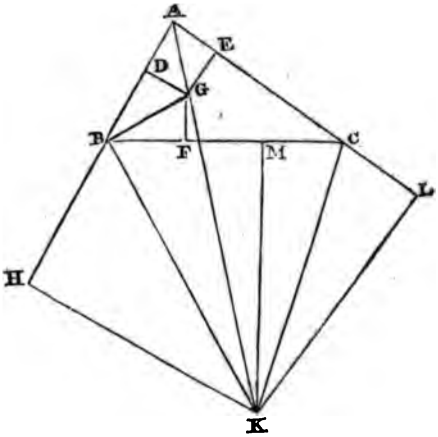


From A, draw AD perpendicular upon the opposite side BC; then (26. & 27. 2.) the difference of the sum of the squares of AB, BC, and the square of the base AC, is equal to twice the rectangle CBD; but twice the rectangle CBA is to twice the rectangle CBD, that is, to the difference of the sum of the squares of AB, BC, and the square of AC, (2. 5.) as AB to BD; that is, by Prop. 1. as radius to the sine of BAD, which is the complement of the angle ABC, that is, as radius to the co-sine of ABC.

PROP. VI.

In any triangle ABC, *whose two sides are* AB, AC, *and base* BC; *the rectangle contained by half the perimeter, and the excess of it above the base* BC, *is to the rectangle contained by the straight lines, by which the half of the perimeter exceeds the other two sides* AB, AC, *as the square of the radius is to the square of the tangent of half the angle* BAC *opposite to the base.*

Let the angles BAC, ABC be bisected by the straight lines AG,



BG; and, producing the side AB, let the exterior angle CBH be bisected by the straight line BK, meeting AG in K; and from the points G, K, let there be drawn perpendicular upon the sides the straight lines GD, GE, GF, KH, KL, KM. Since therefore (2. 6.) G is the centre of the circle inscribed in the triangle ABC, GD, GF, GE will be equal, and AD will be equal to AE, BD to BF, and CE to CF. In like manner KH, KL, KM will be equal, and BH will be equal to BM, and AH to AL, because the angles HBM, HAL are bisected by the straight lines BK, KA: and because in the triangles KCL, KCM, the sides LK, KM are equal, KC is common, and KLC, KMC are right angles, CL will be equal to CM: since therefore BM is equal to BH, and CM to CL; BC will be equal to BH and CL together; and, adding AB and AC together, AB, AC, and BC will together be equal to AH and AL together: but AH, AL are equal: wherefore each of them is equal to half the perimeter of the triangle ABC: but since AD, AE are equal, and BD, BF, and also CE, CF, AB together with FC, will be equal to half the perimeter of the triangle to which AH or AL was shewn to be equal; taking away therefore the common AB, the remainder FC will be equal to the remainder BH: in the same manner is it demonstrated, that BF is equal to CL: and since the points BDGF are in a circle, the angle DGF will be equal to the exterior and opposite angle FBH (14. 3.); wherefore their halves BGD, HBK will therefore be equiangular, and GD will be to BD, as BH to HK, and the rectangle contained by GD, HK will be equal to the rectangle DBH or BFC; but since AH is to HK, as AD to DG, the rectangle HAD (4. 4.) will be to the rectangle contained by HK, DG, or the rectangle BFC, (as the square of AD is to the square of DG, that is) as the square of the radius to the square of the tangent of the angle DAG, that is, the half of BAC: but HA is half the perimeter of the triangle ABC,

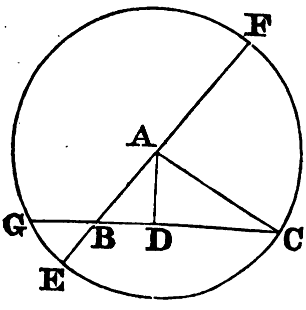
and AD is the excess of the same above HD, that is, above the base BC: but BF or CL is the excess of HA or AL above the side AC, and FC, or HB, is the excess of the same HA above the side AB; therefore the rectangle contained by half the perimeter, and the excess of the same above the base, viz. the rectangle HAD, is to the rectangle contained by the straight lines by which the half of the perimeter exceeds the other two sides, that is, the rectangle BFC, as the square of the radius is to the square of the tangent of half the angle BAC opposite to the base. Q. E. D.

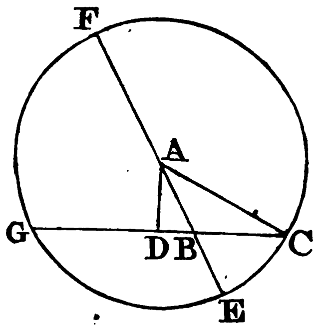
PROP. VII.

In a plane triangle, the base is to the sum of the sides, as the difference of the sides is to the sum or difference of the segments of the base made by the perpendicular upon it from the vertex, according as the square of the greater side is greater or less than the sum of the squares of the lesser side and the base.

Let ABC be a plane triangle; if from A the vertex be drawn a straight line AD perpendicular upon the base BC, the base BC will be to the sum of the sides BA, AC, as the difference of the same sides is to the sum or difference of the segments CD, BD, according as the square of AC the greater side is greater or less than the sum of the squares of the lesser side AB, and the base BC.

About A as a centre, with AC the greater side for a distance, let a circle be described meeting AB produced in E, F, and CB in G:





it is manifest, that FB is the sum, and BE the difference of the sides; and since AD is perpendicular to GC, GD, CD will be equal; consequently GB will be equal to the sum or difference of the segments CD, BD, according as the perpendicular AD meets the base, or the base produced; that is (26. & 27. 2.) according as the square of AC is greater or less than the sum of the squares of AB, BC: but (24. 3.) the rectangle CBG is equal to the rectangle

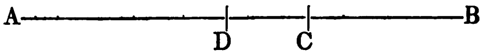
EBF; that is, (16. 5.) BC is to BF, as BE is to BG, that is, the base is to the sum of the sides, as the difference of the sides is to the sum or difference of the segments of the base made by the perpendicular from the vertex, according as the square of the greater side is greater or less than the sum of the squares of the lesser side and the base. Q. E. D.

PROP. VIII. PROB.

The sum and difference of two magnitudes being given, to find them.

Half the given sum added to half the given difference, will be the greater, and half the difference subtracted from half the sum, will be the less.

For, let AB be the given sum, AC the greater, and BC the less. Let AD be half the given sum; and to AD, DB, which are equal,



let DC be added; then AC will be equal to BD and DC together; that is, to BC, and twice DC; consequently twice DC is the difference, and DC half that difference, but AC the greater is equal to AD, DC; that is, to half the sum added to half the difference, and BC the less is equal to the excess of BD, half the sum above DC half the difference. Q. E. D.

SCHOLIUM.

Of the six parts of a plane triangle (the three sides and three angles) any three being given, to find the other three is the business of plane trigonometry; and the several cases of that problem may be resolved by means of the preceding propositions, as in the two following, with the tables annexed. In these, the solution is expressed by a fourth proportional to three given lines; but if the given parts be expressed by numbers from trigonometrical tables, it may be obtained arithmetically by the common Rule of Three.

Note. In the tables the following abbreviations are used: R is put for the Radius; T for Tangent; and S for Sine. Degrees, minutes, seconds, &c. are written in this manner; 30° 25′ 13″, &c. which signifies 30 degrees, 25 minutes, 13 seconds, &c.

SOLUTION of the CASES of right angled TRIANGLES.

GENERAL PROPOSITION.

In a right angled triangle, of the three sides and three angles, any two being given besides the right angle, the other three may be found, except when the two acute angles are given, in which case the ratios of the sides are only given, being the same with the ratios of the sines of the angles opposite to them.

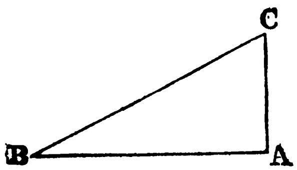
It is manifest from (17. 2.) that of the two sides and hypothenuse any two be given the third may also be found. It is also manifest from (5. 2.) that if one of the acute angles of a right angled triangle be given, the other is also given, for it is the complement of the former to a right angle.

If two angles of any triangle be given, the third is also given, being the supplement of the two given angles to two right angles.

The other cases may be resolved by help of the preceding propositions, as in the following table.

	GIVEN.	SOUGHT.	
1	Two sides, AB AC.	The angles B, C.	AB : AC :: R : T, B, of which C is the complement.
2	AB, BC, a side and the hypothenuse.	The angles B, C.	BC : BA :: R : S, C, of which B is the complement.
3	AB, B, a side and an angle.	The other side AC.	R : T, B :: BA : AC.
4	AB and B, a side and an angle.	The hypothenuse BC.	S, C : R :: BA : BC.
5	BC and B, the hypothenuse and an angle.	The side AC.	R : S, B :: BC : CA.

These five cases are resolved by Prop. 1.

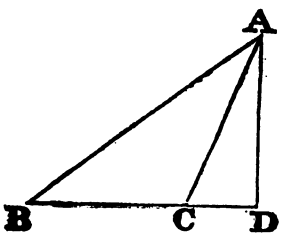


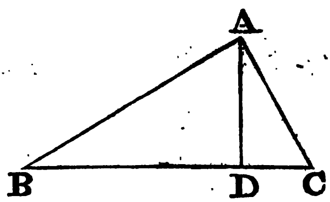
SOLUTION of the CASES of oblique-angled TRIANGLES.

GENERAL PROPOSITION.

In an oblique-angled triangle, of the three sides and three angles, any three being given, the other three may be found, except when the three angles are given; in which case the ratios of the sides are only given, being the same with the ratios of the sines of the angles opposite to them.

	GIVEN.	SOUGHT.	
1	A, B, and therefore C, and the side AB.	BC, AC.	S, C : S, A : : AB : BC, and also S, C : S, B : : AB : AC. (2.)
2	AB, AC, and B, two sides and an angle opposite to one of them.	The angles A and C.	AC : AB : : S, B : S, C. (2.) This case admits of two solutions; for C may be greater or less than a quadrant. (Cor. to Def. 4.)
3	AB, AC, and A, two sides, and the included angle.	The angles B and C.	AB+AC : AB−AC : : T, $\frac{C+B}{2}$: T, $\frac{C-B}{2}$; (3.) and the sum and difference of the angles C, B, being given, each of them is given. (8.) Otherwise. BA : AC : : R : T, ABD, and also R : T, ABD−45° : T, $\frac{B+C}{2}$: T, $\frac{B-C}{2}$; (4.) therefore B and C are given as before. (8.)





	GIVEN.	SOUGHT.	
4	AB, BC, CA, the three sides.	A, B, C, the three angles.	2 AC×CB : ACq+CBq −ABq :: R : Co S, C. If ABq+CBq be greater than ABq. 2 AC×CB : ABq−ACq − CBq :: R : Co S, C. If ABq be greater than ACq+CBq. (4.) Otherwise. Let AB + BC + AC = 2P; $P\times\overline{P-AB} : \overline{P-AC} \times\overline{P-BC}$ :: Rq : Tq, ½ C, and hence C is known. (6). Otherwise. Let AD be perpendicular to BC. 1. If ABq be less than ACq+CBq, BC : BA +AC :: BA−AC : BD−DC, and BC the sum of BD, DC is given; therefore each of them is given. (8.). 2. If ABq be greater than ACq+CBq; BC : BA +AC :: BA−AC : BD+DC, and BC the difference of BD, DC is given: therefore each of them is given. (8.) And CA : CD :: R : Co S, C. (1.) and C being found, A and B are found by case 2. or 3.

THE END.